Black holes

Part 2. Black hole thermodynamics

Dmitry Levkov

INR RAS & ITMP MSU

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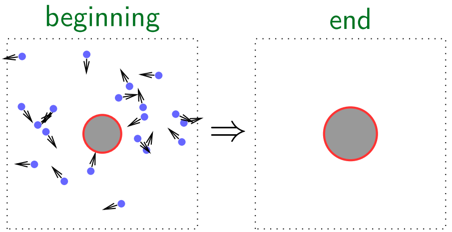
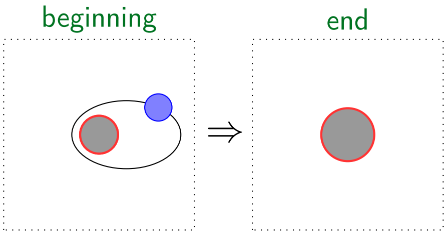
No–hair theorems





Black hole eating particles and a planet

No–hair "theorem" 1

Matter is either eaten by BH
or flies away to infinity

→ proven for weak perturbations around BH

→ no general proof

→ but valid in known cases

No–hair theorem 2

Black hole with mass M, angular momentum a and charge Q is the most general stationary solution in GR

→ proved in GR + electrodynamics

kind of obvious ...

No–hair theorems in a nutshell

Black holes have no hair

No–hair "theorem" 1

Black holes are the end-states of evolution!

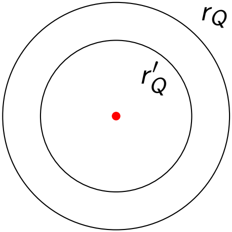
No–hair theorem 2

All black holes are identical ... except for M, a, and Q

People

black holes

Charged black hole



- $a = 0$, $Q \neq 0$
- Charged BH = Reissner-Nordstrom BH
- Interval:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

the same form!

- But: $$f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$$
- Not a vacuum solution: $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$

  Electric potential: $A_0 = Q/r$ $\leftarrow$ creates $T_{\mu\nu} \neq 0$!
- Horizon: $f(r) = 0$ $\leftarrow$ not a singularity! (the same argument)

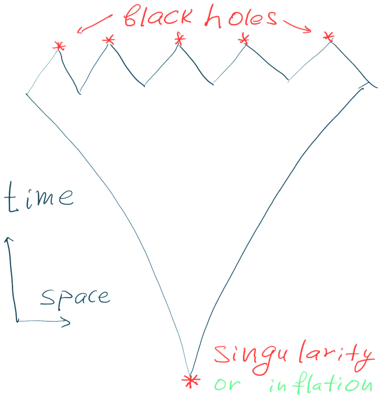
  quadratic eq. $\Rightarrow$ $r_Q = GM\left[1 + \sqrt{1 - \frac{Q^2}{GM^2}}\right]$

  second solution $r'_Q < r_Q$ — <u>under horizon</u>

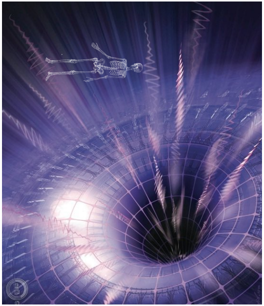
Classical black holes = graveyards in the Universe

With time:

- planets will fall onto stars
- stars will fly away
  or fall into central BHs
- accelerated expansion
  ⇒ outer space will be empty



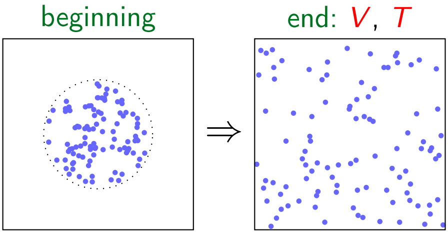
only BHs remain in the dark ...



Classical black holes = perfect matter & information storages

but they do not give it back ...

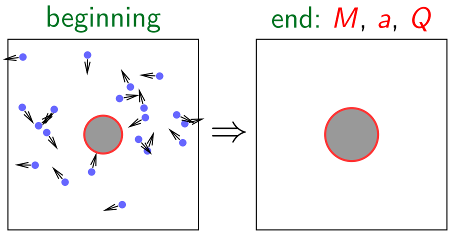
Black holes = thermal equilibria?



gas thermalizes in the box

Thermodynamics

1. With time, complex systems thermalize
2. Thermal equilibria ↔ few macro-parameters



black hole eats particles

Black holes

1. With time, systems collapse into BHs
2. Black holes ↔ M, a, Q

similarity so far ...

Cannot live with classical gravity!

Schrödinger cat

$|\text{dead}\rangle + |\text{alive}\rangle$

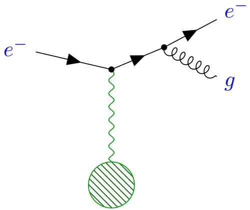
Schrödinger cat in GR

$$|\text{dead}, g_{\text{dead}}\rangle + |\text{alive}, g_{\text{alive}}\rangle$$

$\Rightarrow$ a state of grav. field exists

$\Rightarrow$ sum of grav. states = grav. state

Emission of grav. wave



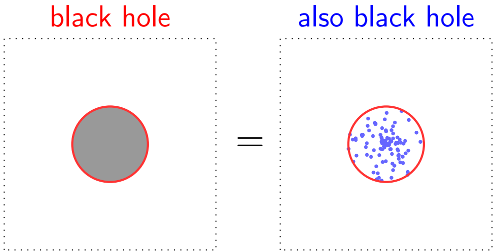
$$|\Psi_{in}\rangle \xrightarrow{\text{evolution}} |\Psi_{out}\rangle$$

$\Rightarrow$ grav. wave consists of gravitons

$\Rightarrow$ metric $\leftrightarrow$ virtual gravitons

Gravity is the ordinary quantum theory!

Quantum black hole



black hole = bound state of gravitons

But we do not know how to quantize it!

Number of gravitons inside black hole

- **Grav. waves:** $g_{\mu\nu} = \eta_{\mu\nu} + \underbrace{h_{\mu\nu}}_{\text{wave}}$

  $\Rightarrow$ GR eqs: $\Box\, h_{ij}^{TT} = 0 \leftarrow$ linearized in h

  $\Rightarrow$ gravitons are massless: $E_g \equiv \omega_g = |\boldsymbol{p}_g|$

- **Uncertainty principle:**

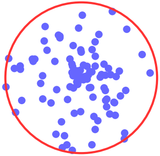
$$E_g \sim p_g \geq \frac{1}{r_h} \equiv \frac{1}{2GM}$$

- $\Rightarrow$ **Number of gravitons inside BH:**

$$N_g \sim \frac{M}{E_g} \lesssim GM^2 \quad \text{or} \quad \boxed{N_g \lesssim \frac{r_h^2}{l_{pl}^2}}$$

  $\rightarrow A_h \equiv = 4\pi r_h^2$ — area of horizon

  $\rightarrow l_{pl} = \sqrt{G} \sim 10^{-33}$ cm — Planck length



black hole: r_h

$N_g \lesssim A_h / l_{pl}^2$

Black holes are two–dimensional?

$N_g \sim (r_h/l_{pl})^2$

Number of gravitons: $N_g \propto r_h^2$

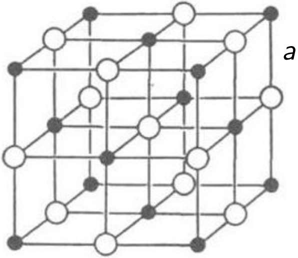
Number of states:

→ each graviton: $\sim k$ states

→ N_g gravitons: $\Gamma \sim k^{N_g}$ states

→ **Entropy:** $S_B \equiv \ln \Gamma \sim \# \frac{r_h^2}{l_{pl}^2}$

$N \sim (L/a)^3$



Number of aroms: $N \propto L^3$

Number of states:

→ each: $\sim k$ states

→ N atoms: $\Gamma \sim k^N$ states

→ **Entropy:** $S \propto L^3$

There is nothing behind the horizon?

Bekenstein bound and quizzical holography

Practical problem: maximal hard-drive storage

- Maximal N_γ inside region R
- Uncertainty:

  $E_\gamma \sim p_\gamma \gtrsim R^{-1}$ or $E \sim N_\gamma E_\gamma \gtrsim \frac{N_\gamma}{R}$
- Hoop conjecture:

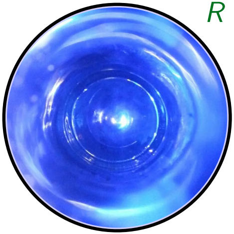
  $R \lesssim 2GE$ or collapse!
- Put everything together:

  $$\Rightarrow N_\gamma \lesssim \frac{R^2}{G} \sim \frac{R^2}{l_{pl}^2} \quad \text{and} \quad \boxed{S \equiv \ln \Gamma \lesssim \frac{R^2}{l_{pl}^2}}$$

  Bekenstein bound
- Cannot pack large entropy into R!

  or it collapses

  Black holes have the maximal entropy



N_γ photons

The entire world in two-dimensional!

Statistics = ensembles of systems

Black holes resemble thermal states...

p_1 $|\Psi_1\rangle$ | p_2 $|\Psi_2\rangle$ | $\dots$ | p_Γ $|\Psi_\Gamma\rangle$

many identical systems

Density matrix:

- $\hat{\rho} = |\Psi\rangle\langle\Psi|$ - one system: $\langle\hat{A}\rangle \equiv \mathrm{tr}(\hat{\rho}\,\hat{A}) = \langle\Psi|\hat{A}|\Psi\rangle$
- $\hat{\rho} = \sum_n \underbrace{p_n}_{\text{probability}} |\Psi_n\rangle\langle\Psi_n|$ - ensemble: $\langle\hat{A}\rangle \equiv \mathrm{tr}(\hat{\rho}\,\hat{A}) = \sum p_n \langle\Psi_n|\hat{A}|\Psi_n\rangle$
- normalization: $\langle 1\rangle = \mathrm{tr}\,\hat{\rho} = 1$

Thermal equilibrium: $\boxed{\hat{\rho} = Z^{-1}\mathrm{e}^{-\hat{H}/T}}$ — Boltzmann exponent

Normalization: $Z = \mathrm{tr}\,\mathrm{e}^{-\hat{H}/T}$

With time, systems arrive into thermal equilibrium

Thermal instantons

Statistical sum: $Z = \mathrm{tr}\, \mathrm{e}^{-\hat{H}/T}$

- Related to evolution operator: $T^{-1} \equiv -it_\beta$

  $$Z = \mathrm{tr}\, \mathrm{e}^{-i\hat{H}t_\beta}$$

  but with imaginary time $t = -i \underbrace{\tau}_{\text{Euclidean time}}$

- Euclidean time: $0 \le \tau \le T^{-1}$
- Path integral:

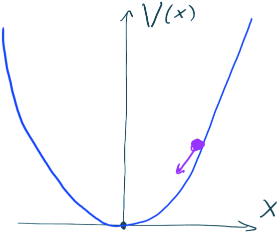
  $$Z = \underbrace{\int dx_0}_{\text{tr}} \underbrace{\int dx(\tau)\Big|_{x_0,\,\tau=T^{-1}}^{x_0,\,\tau=0}}_{\text{periodic trajectories}} \underbrace{\mathrm{e}^{-S_E[x]}}_{\text{cl. action } t \to -i\tau}$$

- Saddle-point method: $S_E \gg 1$

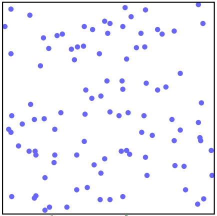
  main contributions $\underbrace{x \sim x_{cl}(\tau)}_{\text{instanton}} : \underbrace{S_E \text{ minimal}}$!

  instanton = class. solution!

- Thermal instantons = periodic in τ solutions



oscillator



thermal gas

result

$Z = \mathrm{e}^{-S_E[x_{cl}]}$

Thermal instantons in quantum gravity

Gibbons, Hawking '77

Consider quantum gravity at temperature T

- We did not even quantize gravity!
- Nevertheless: $t = -i\tau$, $g^E_{\mu\nu} = g^E_{\mu\nu}(\tau, x)$

$$Z = \underbrace{\int dg^E_{\mu\nu}(\tau,\, x)\Big|_{\text{periodic}}}_{\text{period } T^{-1}} \underbrace{\mathrm{e}^{-S_{gr,\,E}[g^E]}}_{grav.\ action}$$

- $S_{gr,\,E} = \frac{1}{16\pi G}\int d^4x^E \sqrt{g^E}\, R + \text{boundary term}$
- Thermal instanton $=$ periodic in τ solution

  We already have stationary ($\Rightarrow$ periodic) solution!
- A black hole: $$\boxed{ds^2 = +f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2}$$
- **But now it is singular!**

  $\boxed{f(r_h) = 0}$ $\leftarrow$ not covered by the horizon!

Look closely!

Gibbons, Hawking '77

$$ds^2 = +f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

- Zoom into horizon: $f \approx (r - r_h) f'_h$
- Introduce

$$\rho = 2\sqrt{\frac{r - r_h}{f'_h}} \quad \text{and} \quad \vartheta = \frac{f'_h}{2}\tau$$

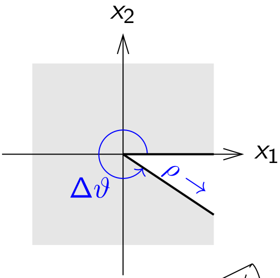
- Obtain $ds^2 = \underbrace{\rho^2 d\vartheta^2 + d\rho^2}_{\text{flat plane!}} + r_h^2 d\Omega^2$

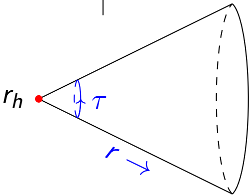
- But $\Delta\vartheta = \frac{f'_h}{2}\Delta\tau = \frac{f'_h}{2T}$

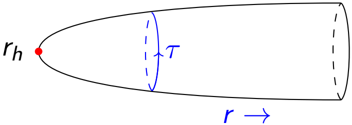
  $\to \Delta\vartheta = 2\pi$ — plane

  $\to \Delta\vartheta \neq 2\pi$ — cone singularity

- $$T_H = \frac{f'_h}{4\pi} = \frac{1}{8\pi GM}$$ **Hawking (BH) temperature**



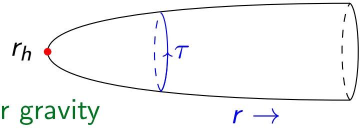




Statistical sum for gravity

$$T_H = \frac{c^3 \hbar}{8\pi k_B G M}$$

Quantum & relativistic thermodynamics for gravity



- Substitute the instanton: $Z = \mathrm{e}^{-S_{gr}^E[g_{BH}^E]} = \mathrm{e}^{-4\pi G M^2}$
- Entropy: imagine that all states have the same energy!

$$Z = \sum_n \mathrm{e}^{-E_n/T_H} = \underbrace{\mathrm{e}^{S_B}}_{\text{number of states}} \cdot \underbrace{\mathrm{e}^{-M/T_H}}_{\text{BH mass}}$$

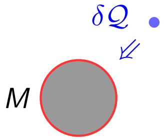
- **Bekenstein entropy:**

$$S_B = \ln Z + \frac{M}{T_H} = 4\pi G M^2 = \frac{A_h}{4 l_{pl}^2}$$

First law of thermodynamics

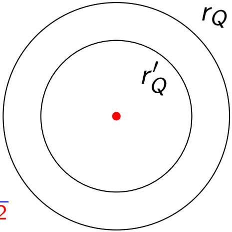
$$T_H dS \equiv \delta Q = dM$$

Automatically satisfied!



First law for charged black hole

$$f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$$



- Horizon: $r_Q = GM\left[1 + \sqrt{1 - \frac{Q^2}{GM^2}}\right]$
- Temperature:

$$T_H = \frac{f'_h}{4\pi} = \frac{1}{2\pi GM}\,\frac{\sqrt{1 - Q^2/GM^2}}{[1 + \sqrt{1 - Q^2/GM^2}]^2}$$

- Entropy: $S_Q = \frac{4\pi r_Q^2}{4 l_{pl^2}}$
- First law: $T_H dS_Q = dM - \frac{Q}{r_Q}\, dQ$ $\leftarrow$ Extra term with $A_0(r_Q) = Q/r$ work to bring dQ!
- **Critical BH:** $M = QM_{pl}$

  $T_H = 0$, but $S_Q \neq 0$!

Estimates

Entropy:

- Entropy of matter in the entire Universe:

$$S_U = \frac{2000}{\text{cm}^3} \times \text{Vol}(30\ \text{Gpc}) \sim\ 10^{90}$$

- Black hole in the Milky Way:

$$S_B = 4\pi G(4 \cdot 10^6\ M_\odot)^2 \sim\ 10^{89}$$

Black holes keep all the entropy!

Temperature:

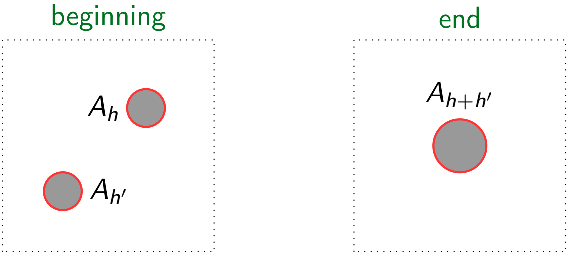
- Black hole in the Milky Way, $M \sim 4 \cdot 10^6\ M_\odot$: $T_H \sim 10^{-14}$ K
- Astrophysical black holes, $M \sim 3\ M_\odot$: $T_H \sim 10^{-8}$ K
- Moon-mass black hole, $M \sim 4 \cdot 10^{-8}\ M_\odot$: $T_H \sim 2$ K
- Asteroid-mass black hole, $M \sim 10^{-12}\ M_\odot$: $T_H \sim 10^4$ K
- Smallest primordial black hole, $M \sim 10^{14}$ g: $T_H \sim 10^{12}$ K
- Planckian black hole, $M \sim M_{pl}$: $T_H \sim M_{pl}$

Small (or not...)

Second law of black hole thermodynamics

Black hole area theorem *Hawking '71*

If GR equations are valid, then:
total area BH horizons grows in the process of evolution.



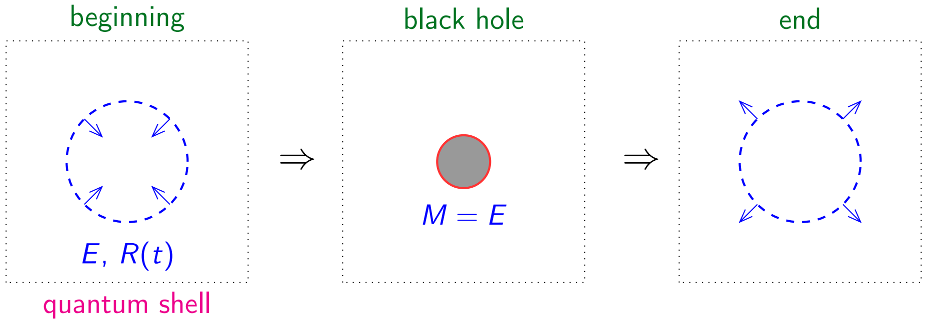
$$A_h + A_{h'} \leq A_{h+h'}$$

Entropy grows ↔ second law!

Black hole entropy from scattering

Bezrukov, DL, Sibiryakov '15

Collapse of a quantum spherical shell:



Calculate it semiclassically!

Result: $P(\text{contraction} \to \text{expansion}) \sim \mathrm{e}^{-\pi E^2/M_{pl}^2} = \mathrm{e}^{-S_B}$

A probability of choosing 1 state out of $\Gamma \sim \mathrm{e}^{S_B}$ states!

Black hole thermodynamics

law №	hot bodies	black holes
0	systems thermalize with time	BHs eat surrounding matter
	equilibrium is characterized by few parameters (gas: V и T)	BHs: mass M, charge Q & angular momentum a
I	energy conservation $\delta\mathcal{Q} \equiv TdS = dE + pdV$	$T_H dS_B = dM - A_0(r_Q)dQ$
II	entropy cannot decrease	total area of all BH horizons cannot decrease
III	entropy is zero at zero temperature	?

Third law of black hole thermodynamics

Critical BH: $M = QM_{pl}$, $T_H = 0$, $S_Q \neq 0$

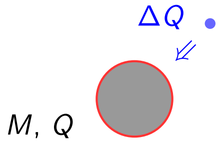
$\Rightarrow$ Entropy is nonzero at zero temperature!

(N.B. Doubts in stability of critical BHs!)

still ...

Alternative (safe) formulation

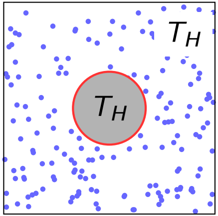
Critical BH cannot be reached in finite time



Particular calculations:

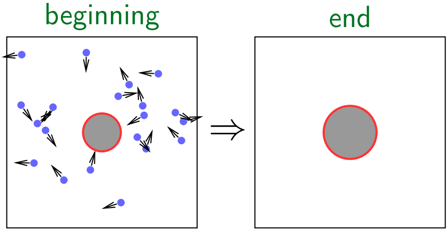
This holds both for black holes and for ordinary systems!

The story is not yet consistent!



thermal equilibrium?

- No, this will happen:



All hot systems emit particles!

... and black holes do (next lecture)!

Summary

Black holes

- Unique solutions with few parameters (have no hair): M, Q, a
- Bound states of gravitons
- Periodic in Euclidean time τ with period T_H^{-1}
  $\Rightarrow$ have temperature $T_H = (8\pi GM)^{-1}$ $\leftarrow$ Hawking temp.
- Have entropy $S_B = A_h/4l_{pl}^2$ $\leftarrow$ Bekenstein entropy
  $\Rightarrow$ they are two-dimensional
  $\rightarrow$ this is the maximal possible entropy (Bekenstein bound)
  $\Rightarrow$ the world is two-dimensional!
- They shine! (next lecture)

Black hole thermodynamics

0. Black holes eat all surrounding matter & they are unique
I. Energy conservation
II. Total area of black hole horizons cannot decrease with time
III. Critical black holes cannot be reached in finite time.

Thank you for attention!