The eigenvalue spectrum of a large real antisymmetric random matrix with non-zero mean

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Goals and objectives

Goal: Finding the spectrum of random matrices

Objectives:

1. A detailed study of the case of a symmetric matrix.
2. Antisymmetric random matrix with zero mean:
   - zero mean value of the matrix element → Wigner semicircle
3. Antisymmetric random matrix with non-zero mean:
   - non-zero mean → spectral superposition + renormalization

Problem formulation

M – symmetric large matrix: $M_{ij} = M_{ji}$, $N \to \infty$.
$M_{ij} \sim \mathcal{N}(0, \sigma^2)$, $\quad M_{ij} \sim \mathcal{N}(\frac{M_0}{N}, \sigma^2)$:

$$p(M_{ij}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{M_{ij}^2}{2\sigma^2}}, \quad p(M_{ij}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(M_{ij} - \frac{M_0}{N}\right)^2}{2\sigma^2}}, \quad (1)$$

where $\sigma^2 = \frac{J^2}{N}$ $(J \sim 1)$.

Spectral density of the matrix

$$\nu(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i). \tag{2}$$

Sokhotski–Plemelj theorem:

$$\lim_{\epsilon \to 0^+} \frac{1}{\lambda - i\epsilon} = i\pi\delta(\lambda) + \mathcal{P}\left(\frac{1}{\lambda}\right). \tag{3}$$

Linear algebra

$$\det(\mathbf{I}\lambda - \mathbf{M}) = \prod_{i=1}^{N} (\lambda - \lambda_i), \tag{4}$$

$$\nu(\lambda) = \frac{1}{\pi N} \mathrm{Im} \frac{\partial}{\partial \lambda} \log(\det(\mathbf{I}(\lambda - i\epsilon) - \mathbf{M})). \tag{5}$$

Replica trick

$$\log x = \lim_{n\to 0} \frac{1}{n}(x^n - 1), \tag{6}$$

where n is assumed to be an integer in the equations, then we go to the limit $n \to 0$.

$$\nu(\lambda) = -\frac{2}{\pi N} \mathrm{Im} \frac{\partial}{\partial \lambda} \lim_{n\to 0} \frac{1}{n} [(\det^{-\frac{1}{2}}(\mathbf{I}\lambda_\epsilon - \mathbf{M}))^n - 1], \tag{7}$$

where $\lambda_\epsilon \equiv \lambda - i\epsilon$.

Master formula

The Gaussian integral is related to the determinant:

$$\det^{-\frac{1}{2}}(\mathbf{I}\lambda_\epsilon - \mathbf{M}) = \left(\frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi}}\right)^N \int_{-\infty}^{+\infty} \prod_i dx_i \exp\left(-i\sum_{i,j;\alpha} x_i^\alpha(\lambda\delta_{ij} - M_{ij})x_j^\alpha\right) \quad (8)$$

$$\nu(\lambda) = -\frac{2}{\pi N}\mathrm{Im}\frac{\partial}{\partial\lambda}\lim_{n\to 0}\frac{1}{n}\left[\left(\frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi}}\right)^{Nn}\int_{-\infty}^{+\infty}\prod_i dx_i \times \times \exp\left(-i\sum_{i,j;\alpha} x_i^\alpha(\lambda\delta_{ij} - M_{ij})x_j^\alpha\right) - 1\right] \quad (9)$$

Probability density

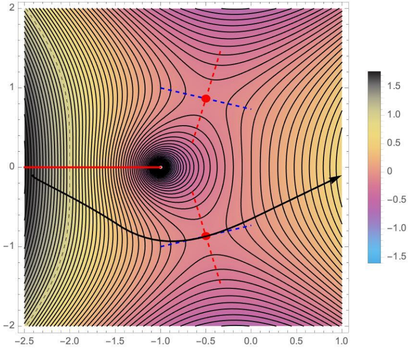
$M_{ij} \sim \mathcal{N}(0, \sigma^2), \quad M_{ij} \sim \mathcal{N}(\frac{M_0}{N}, \sigma^2)$:

$$p(M_{ij}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{M_{ij}^2}{2\sigma^2}}, \quad p(M_{ij}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(M_{ij} - \frac{M_0}{N}\right)^2}{2\sigma^2}}, \quad (10)$$

where $\sigma^2 = \frac{J^2}{N}$ $(J \sim 1)$.
Average spectrum density

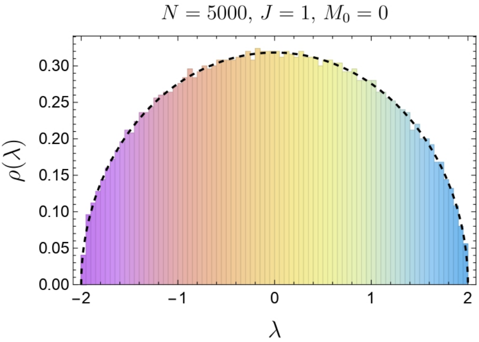
$$\rho_0(\lambda) = \langle \nu(\lambda) \rangle_{M_{ij}} = \int \nu(\lambda; \{M_{ij}\}) \prod_{i<j} p(M_{ij})\, dM_{ij}. \quad (11)$$

Complex analysis



$$\int_{-\infty}^{+\infty} ds \exp\left(-N g(s)\right), \quad g(s) = \frac{\lambda^2 s^2}{4J^2} + \frac{1}{2}\log(i(1+s)). \qquad (12)$$

Wigner semicircle



$$\rho_0(\lambda) = \begin{cases} \frac{\sqrt{4J^2 - \lambda^2}}{2\pi J^2}, & |\lambda| < 2J; \\ 0, & |\lambda| > 2J. \end{cases} \tag{13}$$

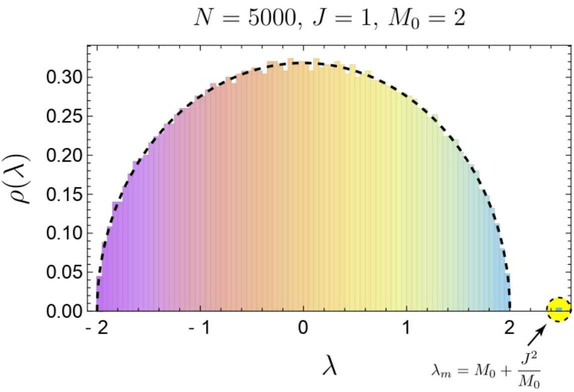
$M_{ij} \sim \mathcal{N}(\frac{M_0}{N}, \sigma^2)$:

$$p(M_{ij}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(M_{ij} - \frac{M_0}{N}\right)^2}{2\sigma^2}}, \quad (14)$$

where $\sigma^2 = \frac{J^2}{N}$ $(J \sim 1)$.
Integral in case of non-zero mean

$$\int_{-\infty}^{+\infty} ds \, \frac{(1+s)^{\frac{1}{2}}}{[i(s_1 - s)]^{\frac{1}{2}}} e^{-Ng(s)}, \quad s_1 = -1 + \frac{M_0}{\lambda}. \quad (15)$$

Non-zero mean



$$\rho_{J_0}(\lambda) = \begin{cases} \rho_0(\lambda) + \delta\left(\lambda - \left(M_0 + \frac{J^2}{M_0}\right)\right), & |M_0| > J; \\ \rho_0(\lambda), & |M_0| < J. \end{cases} \tag{16}$$

Non-zero antisymmetric matrix

J – antisymmetric matrix: $J_{ij} = -J_{ji}$ with eigenvalues $\pm i\lambda_i$. N – even number.

$$\nu(\lambda) = \frac{1}{N}\sum_{i=1}^{N/2}(\delta(\lambda-\lambda_i)+\delta(\lambda+\lambda_i)). \tag{17}$$

$$\det(\mathbf{I}\lambda_\epsilon - i\mathbf{J}) = \int \prod_{i=1}^{N} d\bar{c}_i dc_i e^{-\sum\limits_{i,j}\bar{c}_i(\mathbf{I}\lambda_\epsilon - i\mathbf{J})_{ij}c_j}, \tag{18}$$

$$\nu(\lambda) = \frac{1}{\pi N}\mathrm{Im}\frac{\partial}{\partial\lambda}\lim_{n\to 0}\frac{1}{n}\left[\int\prod_{i;\alpha} d\bar{c}_i^\alpha dc_i^\alpha e^{-\sum\limits_{i,j}\bar{c}_i^\alpha(\mathbf{I}\lambda_\epsilon - i\mathbf{J})_{ij}c_j^\alpha} - 1\right] \tag{19}$$

Main result

- Reproduced the Wigner semicircle for the zero-mean
-

$$\rho_{J_0}(\lambda) = \rho_0(\lambda) + \frac{1}{N} \sum_{k \in \{j : \lambda_j^* > J\}} \delta(\lambda - \mu_k) + \delta(\lambda + \mu_k), \tag{20}$$

where $\lambda_k^* = \frac{J_0}{N}\lambda_k$ and $\mu_k = \lambda_k^* + \frac{J^2}{\lambda_k^*}$.

Main result

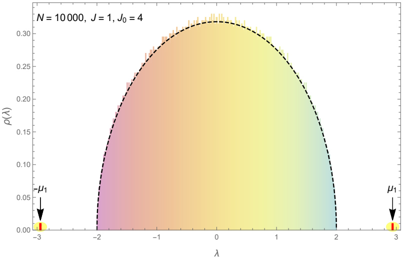
More comprehensible way

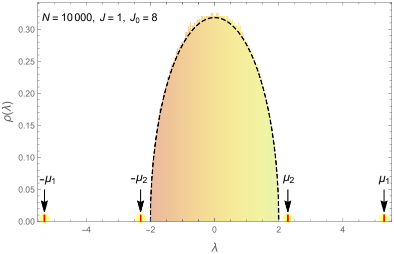
$$\rho_{J_0}(\lambda) = \begin{cases} \rho_0(\lambda), & J_0 < J_{0,1}^{cr}\,; \\ \rho_0(\lambda) + \frac{1}{N}(\delta(\lambda-\mu_1)+\delta(\lambda+\mu_1)), & J_{0,1}^{cr} < J_0 < J_{0,2}^{cr}\,; \\ \dots & \\ \rho_0(\lambda) + \frac{1}{N}\sum_{k=1}^{N/2}\delta(\lambda-\mu_k)+\delta\left(\lambda+\mu_k\right), & J_{0,\frac{N}{2}}^{cr} < J_0\,, \end{cases} \tag{21}$$

where $J_{0,k}^{cr} = \frac{JN}{\lambda_k}$, $k \in \{1,\dots,N/2\}$. When the value of J_0 passes the critical value $J_{0,k}^{cr}$, a two new delta peaks $\delta(\lambda \pm \mu_k)$ appear.

Antisymmetric matrix

Spectral density in the case of a nonzero mean:





What's the result?

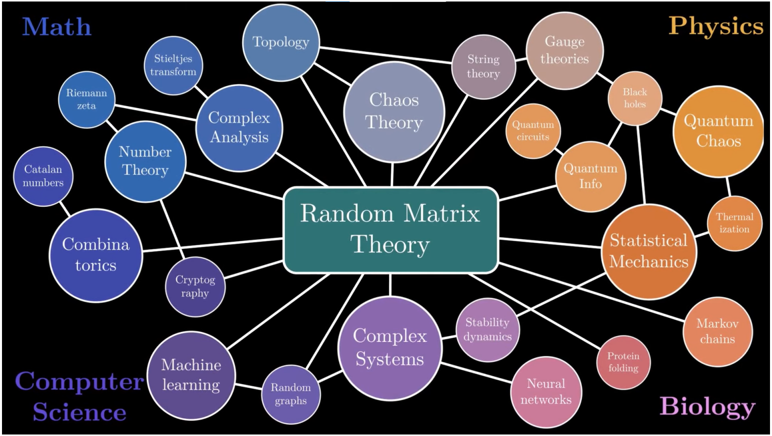
- We have solved the case of a symmetric matrix (this is the answer known to science).
- Were able to generalise to the antisymmetric case.

What's next?

- Sachdev-Ye-Kitaev (SYK) quantum model with a disorder that has a non-zero mean.
- Wigner surmise (the density of distances between adjacent energy levels).

Thank for you attention!

Significance of the problem



Grassmannian variables

$$\{\eta, \theta\} = \eta \cdot \theta + \theta \cdot \eta = 0 \Rightarrow \theta^2 = 0, \tag{22}$$

$$\int d\theta \; a = 0, \quad \int d\theta \; b\theta = b, \tag{23}$$

$$\int d\theta = \frac{\partial}{\partial \theta}. \tag{24}$$

Habbard-Stratanovich transform

Based on the completing a square

$$\exp\left\{-\frac{a}{2}x^2\right\} = \sqrt{\frac{1}{2\pi a}} \int\limits_{-\infty}^{+\infty} \exp\left[-\frac{s^2}{2a} - ixs\right] ds, \quad a > 0 \qquad (25)$$

For each variable $x_i x_j$, we need to introduce an additional integration variable S_{ij}

SYK model

Let n be an integer and m an even integer such that $2 \leq m \leq n$, and consider a set of Majorana fermions $\psi_1, \ldots, \psi_n$ which are fermion operators satisfying conditions:

1. Hermitian $\psi_i^\dagger = \psi_i$;
2. Clifford relation $\{\psi_i, \psi_j\} = 2\delta_{ij}$.

Let $J_{i_1 i_2 \cdots i_m}$ be random variables whose expectations satisfy

$$\langle J_{i_1 i_2 \cdots i_m} \rangle = 0, \quad \langle J^2_{i_1 i_2 \cdots i_m} \rangle = \frac{\sigma^2}{N^{m-1}} \tag{26}$$

Then the SYK$_m$ model is defined as

$$H_{\mathsf{SYK}_m} = \frac{i^{m/2}}{m!} \sum_{1 \leq i_1 < \cdots < i_m \leq n} J_{i_1 i_2 \cdots i_m} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_m} \tag{27}$$

Wigner surmise