Nonequillibrium Schwinger-Keldysh correlators and its analytic properties

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Abstract

We develop Schwinger-Keldysh in-in formalism for generic nonequilibrium dynamical systems with mixed initial states. We construct the generating functional of in-in Green's functions and expectation values for a generic density matrix of the Gaussian type and show that the requirement of particle interpretation selects a distinguished set of positive/negative frequency basis functions of the wave operator of the theory, which is determined by the density matrix parameters. Then we consider a special case of the density matrix determined by the Euclidean path integral of the theory, and show that its Wightman Green's function satisfy Kubo-Martin-Schwinger quasiperiodicity conditions which hold despite the nonequilibrium nature of the physical setup.

Can correlation functions of nonequillibrium field theory obey Kubo-Martin-Schwinger condition $\langle\hat\phi(t-i\beta)\hat\phi(t')\rangle_\beta = \langle\phi(t')\phi(t)\rangle_\beta$?

Gaussian field theory

We begin with most general Gaussian field theory

$$S[\phi] = \frac{1}{2}\int dt\Big(\dot\phi^T A\dot\phi + \dot\phi^T B\phi + \phi^T B^T\dot\phi + \phi^T C\phi\Big)$$

where ϕ^I are fields, $I = (\boldsymbol{x}, i)$ is multi-index, and $A = A_{IJ}$, $B = B_{IJ}$, $C = C_{IJ}$ are time-dependent operators.

E.o.m. are given by wave operator

$$F = -\frac{d}{dt}A\frac{d}{dt} - \frac{d}{dt}B + B^T\frac{d}{dt} + C,$$

obtained from Hessian $F\delta(t-t') = \delta^2 S[\phi]/\delta\phi(t)\,\delta\phi(t')$.

Density matrix, defining the state of the system is given in coordinate space $\langle\varphi_+|\,\hat\rho\,|\varphi_-\rangle = \rho(\varphi_+,\varphi_-)$ and also has Gaussian form

$$\rho(\boldsymbol\varphi) = \frac{1}{Z}\exp\left\{-\frac{1}{2}\boldsymbol\varphi^T\boldsymbol\Omega\,\boldsymbol\varphi + \boldsymbol j^T\boldsymbol\varphi\right\}, \quad \boldsymbol\Omega = \begin{bmatrix} R & S \\ S^* & R^* \end{bmatrix} \quad \boldsymbol j = \begin{bmatrix} j_+ \\ j_- \end{bmatrix}, \tag{1}$$

where $\boldsymbol\varphi = \big[\varphi_+,\ \varphi_-\big]^T$, and source $\boldsymbol j$ allows to incorporate non-Gaussianities to the state.

Generating functional and Green's functions

To examine in-in correlation functions, we calculate generating functional

$$Z[J_1, J_2] = \mathrm{tr}\left[\hat U_{J_1}(T,0)\,\hat\rho\,\hat U^\dagger_{-J_2}(T,0)\right].$$

where $\hat\rho$ is density matrix, and $\hat U_J(T,0)$ is evolution operator with Hamiltonian modified by source term $-J^T(t)\phi(t)$. Generating functional is calculated in path integral formalism and reads

$$Z[\boldsymbol J] = \text{const}\times\exp\left\{-\frac{i}{2}\int_0^T dt\,dt'\,\boldsymbol J^T(t)\boldsymbol G(t,t')\boldsymbol J(t) - \int_0^T dt\,\boldsymbol J^T(t)\boldsymbol G(t,0)\,\boldsymbol j + \frac{i}{2}\boldsymbol j^T\boldsymbol G(0,0)\,\boldsymbol j\right\},$$

where $\boldsymbol J = [J_1,\ J_2]^T$. Blocks of matrix Green's function

$$\boldsymbol G(t,t') = \begin{bmatrix} G_{\mathrm T}(t,t') & G_<(t,t') \\ G_>(t,t') & G_{\bar{\mathrm T}}(t,t') \end{bmatrix},$$

contains Feynman, anti-Feynman and Wightmann Green's functions.

Explicit form of $\boldsymbol G(t,t')$ is obtained by solving boundary problem, depending on the state, and can be expressed in terms of state-independent basis functions defining Heisenberg field operators

$$\phi(t) = v(t)\,\hat a + v^*(t)\,\hat a^\dagger$$

subject to Neumann boundary problem

$$Fv(t) = 0, \qquad (iW-\omega)v(t)\big|_{t=0} = 0, \quad (iW+\omega^*)v^*(t)\big|_{t=0} = 0,$$

where $W = A\frac{d}{dt} + B$, and matrix ω is the free parameter. In terms of non-anomalous and anomalous averages

$$\nu = \mathrm{tr}\big[\hat\rho\,\hat a^\dagger\hat a\big], \qquad \kappa = \mathrm{tr}\big[\hat\rho\,\hat a\,\hat a\big],$$

e.g. Wightmann Green's function reads

$$G_>(t,t') = v(t)\,(\nu+I)\,v^\dagger(t') + v^*(t)\,\nu\,v^T(t') + v(t)\,\kappa\,v^T(t') + v^*(t)\,\kappa\,v^\dagger(t') \tag{2}$$

Now, we ask whether it is possible to choose matrix ω such that the anomalous average $\kappa = 0$. This gives the equation on ω, whose solution reads

$$\omega = R^{1/2}\sqrt{I-\sigma^2}R^{1/2}, \quad \sigma \equiv R^{-1/2}SR^{-1/2}. \tag{3}$$

so that the blocks of $\boldsymbol G(t,t')$ are significantly simplified, e.g. the second line in (2) vanishes. Such form is well-known in the context if thermofield dynamics.

Euclidean density matrix

In the context of quantum cosmology it is natural to define the density matrix as the Euclidean path integral, i.e. the state is dynamically described by the system itself

$$\rho_E(\varphi_+,\varphi_-;J_E) = \frac{1}{Z}\times\int\limits_{\phi(\tau_\pm)=\varphi_\pm} D\phi\,\exp\left\{-S_E[\phi] - \int_{\tau_-}^{\tau_+} d\tau\,J_E(\tau)\phi(\tau)\right\},$$

where the Euclidean action is obtained by the Wick rotation

$$iS[\phi(t)]\big|_{t=-i\tau} = -S_E[\phi_E(\tau)].$$

The operator coefficients of Euclidean action defined as

$$A_E(\tau) = A(-i\tau), \qquad B_E(\tau) = -iB(-i\tau), \qquad C_E(\tau) = -C(-i\tau),$$

and should satisfy the following quasi-periodicity conditions

$$A_E(\beta-\tau) = A_E^*(\tau), \quad B_E(\beta-\tau) = -B_E^*(\tau), \qquad C_E(\beta-\tau) = C_E^*(\tau), \tag{4}$$

for some $\beta = \tau_+ - \tau_-$, for the density matrix to be Hermitian. After functional integration Euclidean density matrix has Gaussian form (1) with $R = R_E$, $S = S_E$ defined by boundaty-to-boundary Euclicean Green's function and its derivatives.

KMS condition

It is possible to extend the range of τ such that Euclidean e.o.m. become β-periodic. In this case, according to gerenal Floquet theory, the Euclidean basis functions $u_-(\tau)$, $u_+(\tau)$ obey monodromy relation

$$\Big[u_-(\tau+\beta)\ \ u_+(\tau+\beta)\Big] = \Big[u_-(\tau)\ \ u_+(\tau)\Big]\begin{bmatrix} M_{--} & M_{-+} \\ M_{+-} & M_{++} \end{bmatrix} \tag{5}$$

Imposing Neumann boundary conditions on $u_-(\tau)$, $u_+(\tau)$

$$(W_E-\omega)u_-(\tau)\big|_{\tau=0} = 0, \qquad (W_E+\omega^*)u_+(\tau)\big|_{\tau=\beta} = 0,$$

and demanding that off-diagonal blocks vanish $M_{+-} = M_{-+} = 0$ we find that this condition is fulfilled by ω exactly the same as in (3)! This, we have the following monodromy relation

$$u_-(t+\beta) = u_-(\tau)\frac{\nu+I}{\nu}, \qquad u_+(t+\beta) = u_+(t)\frac{\nu}{\nu+I}.$$

With this choice of ω both for Euclidean and Loranzian basis functions are the analytic continuation of each other. Together with the monodromy relation for $u_-(\tau)$, $u_+(\tau)$ this implies

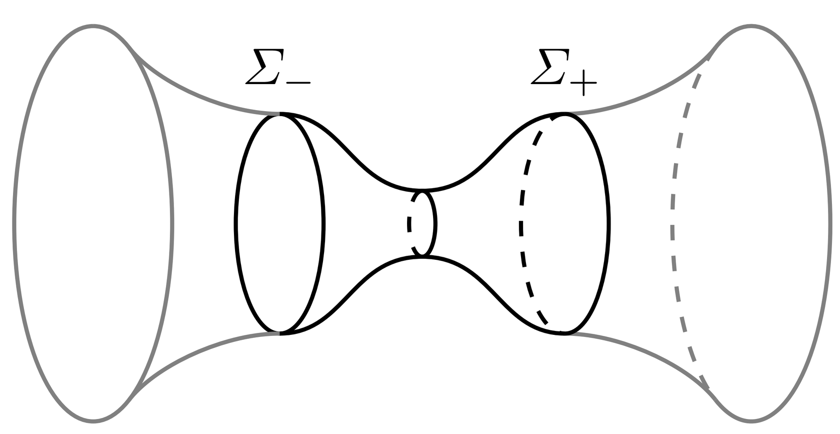
$$v(t-i\beta) = v(t)\frac{\nu+I}{\nu}, \qquad v^*(t-i\beta) = v^*(t)\frac{\nu}{\nu+I}.$$

Using this property in (2), we obtain

$$G_>(t-i\beta,t') = G_<(t,t'),$$

which is nothing but the Kubo-Martin-Schwinger condition. Hence, the KMS condition can hold even in for nonequillibrium field theories!

The conditions (4) under which KMS condition is satisfied may look rather restrictive. However, these are fulfilled at least for the perturbations about the cosmological instanton solution



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