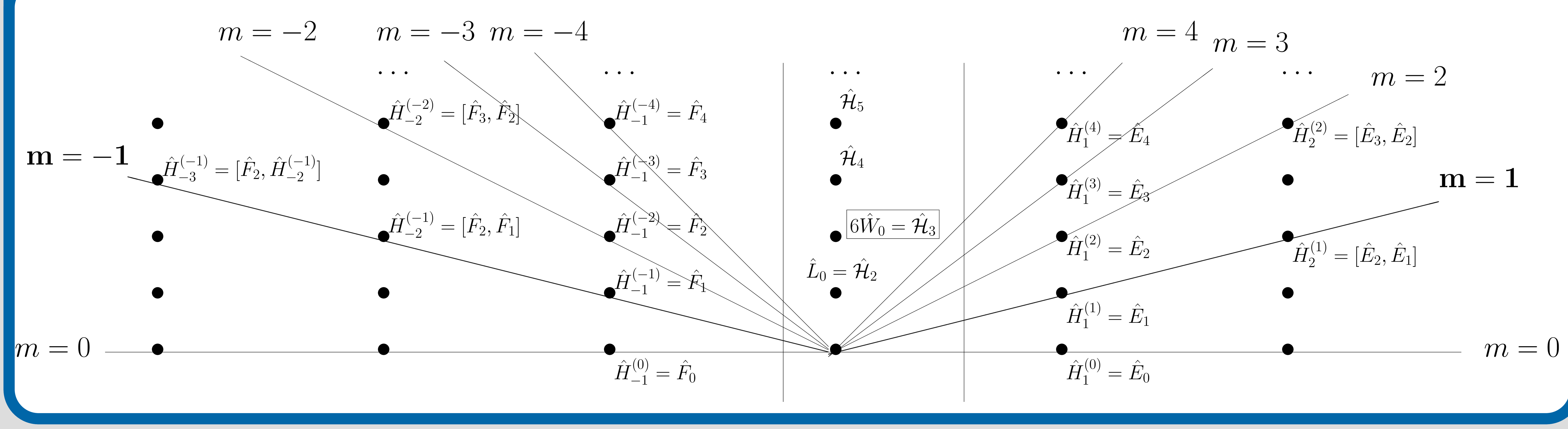
$W_{1+\infty}$ and $\widetilde{W}$ algebras, and Ward Identities

Ya. Drachov[1], A. Mironov[2,3,4], and A. Popolitov[1,3,4]

[1]MIPT, Dolgoprudny, 141701, Russia
[2]Lebedev Physics Institute, Moscow 119991, Russia
[3]NRC "Kurchatov Institute", 123182, Moscow, Russia
[4]Institute for Information Transmission Problems, Moscow 127994, Russia

It was demonstrated recently that the $W_{1+\infty}$ algebra contains commutative subalgebras associated with all integer slope rays (including the vertical one). In our work, we realize that every element of such a ray is associated with a generalized $\widetilde{W}$ algebra. In particular, the simplest commutative subalgebra associated with the rational Calogero Hamiltonians is associated with the $\widetilde{W}$ algebras studied earlier. We suggest a definition of the generalized $\widetilde{W}$ algebra as differential operators in variables p_k basing on the matrix realization of the $W_{1+\infty}$ algebra, and also suggest an unambiguous recursive definition, which, however, involves more elements of the $W_{1+\infty}$ algebra than is contained in its commutative subalgebras. The positive integer rays are associated with $\widetilde{W}$ algebras that form sets of Ward identities for the WLZZ matrix models, while the vertical ray associated with the trigonometric Calogero-Sutherland model describes the hypergeometric τ-functions corresponding to the completed cycles.

Commutative families (integer rays) on the lattice of generators of the $W_{1+\infty}$



The old $\widetilde{W}^{(\pm,n)}$ and $m = \pm 1$ rays

The two families of $\widetilde{W}$ algebras, corresponding to the elements of $m = \pm 1$ commutative subalgebras of $W_{1+\infty}$

$$H^{(1)}_n = \mathrm{tr}\left(\Lambda\frac{\partial}{\partial\Lambda}\Lambda\right)^n = \sum_k p_k \widetilde{W}^{(-,n)}_{k-n}, \qquad (1)$$

$$H^{(-1)}_{-n} = \mathrm{tr}\left(\frac{\partial}{\partial\Lambda}\right)^n = \sum_k p_k \widetilde{W}^{(+,n)}_{k+n}, \qquad (2)$$

where Λ is a $N \times N$ matrix. These $\widetilde{W}$ operators posess explicit recursive definition and by definition are key constituents in Ward identities

$$\widetilde{W}^{(-,n)}_k Z_n = (n+k)\frac{\partial Z_n}{\partial p_{n+k}} \qquad (3)$$

for multi-matrix models

$$Z_n = \iint_{N\times N} dX dY e^{-\mathrm{tr}\,XY + \sum_k \frac{p_k}{k}\mathrm{tr}\,X^k + \frac{1}{n}\mathrm{tr}\,Y^n} \qquad (4)$$

The new $\widetilde{W}^{(m,n)}$ and all integer rays

There is a generalization of (1) and (2) to the whole set of commutative families of $W_{1+\infty}$

$$H^{(m)}_n = \mathrm{tr}\left(\left(\Lambda\frac{\partial}{\partial\Lambda}\right)^m \Lambda\right)^n = \sum_k p_k \widetilde{W}^{(m,n)}_{k-n}, \qquad (5)$$

$$H^{(-m)}_{-n} = \mathrm{tr}\left(\Lambda^{-1}\left(\Lambda\frac{\partial}{\partial\Lambda}\right)^m\right)^n = \sum_k p_k \widetilde{W}^{(-m,-n)}_{k+n}, \qquad (6)$$

which, in turn, also posess recursive defintion and define constraints on generalized WLZZ matrix models

$$Z^{(m)}_n = e^{\frac{1}{n}H^{(m)}_n} \cdot 1 \qquad (7)$$

by Ward identities

$$\widetilde{W}^{(m,n)}_k Z^{(m)}_n = (n+k)\frac{\partial Z^{(m)}_n}{\partial p_{n+k}}. \qquad (8)$$

e-mail drachov.yai@phystech.edu

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