Modern Trends in Mathematical Physics. III

(for pedestrians ... cyclists and drivers)

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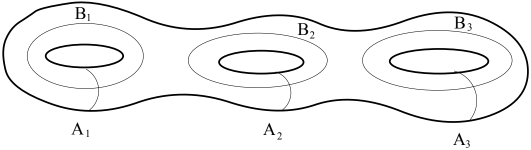
Complex curves

Riemann surface ($g = 3$) Σ:



Smooth Riemann surface (of genus 3)
with marked A- and B-cycles.

Lattice of charges $\Leftrightarrow$ $H_1(\Sigma)$ with symplectic $\langle , \rangle$, $\langle A_i, B_j \rangle = \delta_{ij}$.

Period matrix: $\mathrm{Im}\, T_{ij} \geq 0$, $T \underset{\text{degeneration}}{\rightarrow} \log a$

Period matrices

Computation of elliptic integrals $\oint \lambda \frac{dw}{w}$ or $\oint \frac{dw}{w\lambda}$ on $\Sigma: \ w + \frac{\Lambda^4}{w} = \lambda^2 - U$

At $\Lambda \to 0$, for $U = a^2$

$$w = \lambda^2 - a^2 \qquad (1)$$

so that

$$\oint \lambda \frac{dw}{w} = \oint \lambda d \log\left(\lambda^2 - a^2\right) \sim \left\{ \begin{array}{c} a \\ a \log a - a \end{array} \right.$$

The period "matrix" $\tau \sim \frac{\partial}{\partial a}(a \log a - a) = \log a$, or

$$\tau \sim \int_{-a}^{a} d \log\left(\lambda^2 - a^2\right) \sim \log a \qquad (2)$$

- To be identified with complexified gauge coupling $\tau \sim \frac{\vartheta}{2\pi} + i\frac{4\pi^2}{g^2}$;
- Any parallels with QFT?

Supersymmetric gauge theory

$[\Phi, \Phi^\dagger] = 0$

4d $\mathcal{N} = 2$ supersymmetric Yang-Mills theory:

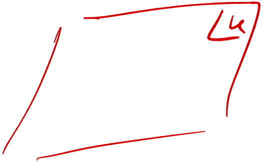
$$\mathcal{L}_0 = \frac{1}{g_0^2}\mathrm{Tr}\ \left(\mathrm{F}_{\mu\nu}^2 + |D_\mu \Phi|^2 + [\Phi, \Phi^\dagger]^2 + \mathrm{fermions}\right) + \frac{\vartheta_0}{2\pi}\mathrm{Tr} F \wedge F \tag{3}$$

Coulomb branch

- Higgs condensate $\langle \Phi \rangle$ breaks gauge group to Abelian;
- Effective $U(1)^{\mathrm{rank}G}$ Abelian theory in IR;
- *Moduli space* of the Coulomb branch: $u \sim \langle \mathrm{Tr}\Phi^2 \rangle$, generally

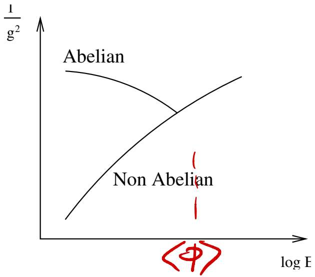
$$P(\lambda; \vec{u}) = \langle \det(\lambda - \Phi) \rangle \tag{4}$$

- 'Light' u-dependent BPS-spectrum ...



Supersymmetry and loop corrections

- Gauge coupling $\frac{1}{g^2} \sim \beta \log \frac{\sqrt{|u|}}{\Lambda}$: *exact* 1-loop RG formula;
- $\beta = 2N - N_f \geq 0$ ... UV completion (?!);



- Complexification: $i\frac{4\pi^2}{g^2} + \frac{\vartheta}{2\pi} = \tau \sim \log \frac{\sqrt{u}}{\Lambda}$;
- Works at $u \gg \Lambda^2$,

SW theory: strong coupling

Obstruction: at $|u| < \Lambda^2$ e.g. one gets $\frac{1}{g^2} \sim \log \frac{\sqrt{|u|}}{\Lambda} < 0$

- Perturbative contributions do not saturate results at strong-coupling;
- Smth else: e.g. instanton contributions ...

Instead ... $\frac{1}{g^2} \sim \mathrm{Im}\tau(\geq 0)$ solves the problem ... together with exact formulas for the period integrals

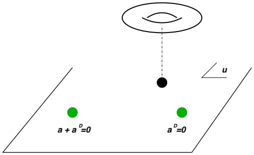
$$\gamma \in H_1(\Sigma) \mapsto a_\gamma(u) = \oint_\gamma \lambda \frac{dw}{w}$$

Already checked that at weak coupling: $a(u) \sim \sqrt{u}$ and $a_D(u) \sim \sqrt{u} \log u \sim \frac{a}{g^2}$ ...

SW theory: strong coupling

Classically at $u = 0$ non-Abelian symmetry restores, but quasi-classical analysis works only at $u \gg \Lambda^2$ ...

Quantum moduli space for $G = SU(2)$



- non-Abelian symmetry *never* restores;
- Instead at $a_D = 0$ and $a + a_D = 0$ Σ degenerates: EM-dual Abelian theory;
- Effective couplings in $\mathcal{N} = 2$ special Kähler geometry: holomorphic prepotential $T_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}$ (action $\mathrm{Im} \int d^4\theta \mathcal{F}(\Phi)$).

Arguments

- Consistency!
- Supported by instanton computations, though:
  - Long story, many attemps through 90-s;
  - Example with finite perturbative renormalization in $SU(2)$ $N_f = 4$ theory;
  - Success with deformation: IR regularization consistent with SUSY, $\Omega(\varepsilon_1, \varepsilon_2)$-background ...
- New formulas valid for:
  - 2d conformal field theory;
  - Solutions of Painlevé equations (isomonodromic deformations).
- Continues beyond validity range of standard QFT methods ...
- 2d CFT and ... differential equations ...

Prepotential expansions

Weak coupling:

$\tau = \frac{\partial^2 \mathcal{F}}{\partial a^2}$

$$\mathcal{F}(a) \underset{a\to\infty}{\to} \frac{1}{2}a^2 \log\frac{a}{\Lambda} + a^2 \sum_{k>0} f_k \left(\frac{\Lambda}{a}\right)^{4k}$$

- Logarithm from $\mathcal{N}=2$ one loop;
- Expansion over instantons of charge k, in powers of $\Lambda^\beta = \Lambda^{2N} = \Lambda^4$: a way to compute $\{f_k\}$ – instanton calculus ...

Strong coupling (monopole point $a_D \to 0$), for $\mathcal{F}_D = aa_D - \mathcal{F}$

$$\mathcal{F}_D(a_D) \underset{a_D\to 0}{\to} -\frac{1}{2}a_D^2 \log\frac{a_D}{\Lambda} - 8\Lambda a_D + a_D^2 \sum_{k>0} f_k^D \left(\frac{a_D}{\Lambda}\right)^{k}$$

Different powers: no instantons in monopole theory! No way to compute $\{f_k^D\}$ other, than to solve a Painlevé equation ...

SU(2)/Painlevé

- Deautonomization: integrable (or isospectral) $\Rightarrow$ isomonodromic system;
- 'SW Toda' (physical pendulum) $\Rightarrow$ Painlevé III

$$\frac{d^2 q}{d\tau^2} + e^{2\tau} \sinh q = 0 \tag{5}$$

- In conventional "isomonodromic" variables ($t \sim \Lambda^4$, $w \sim \sqrt{t} e^q$)

$$H(w, w'; t) = \frac{t w'^2}{4w^2} + \frac{w}{t} + \frac{1}{w} = \partial_t \log \mathcal{T}(t), \tag{6}$$

and ($w_1 = t/w$)

$$w(t)^{-1} = \partial_t t \partial_t \log \mathcal{T}(t) = -t^{1/2} \frac{\mathcal{T}_1(t)^2}{\mathcal{T}(t)^2} \tag{7}$$

Tau-functions

" $1 + \# t + . t^2$

The isomonodromic tau functions ($\epsilon = \emptyset, 1$)

$$\mathcal{T}_\epsilon(t; a, \eta) \underset{t \to 0}{=} \sum_{n \in \mathbb{Z} + \epsilon/2} e^{4\pi i n \eta} t^{(a+n)^2} \frac{\mathcal{B}(a+n, t)}{G(1+2(a+n))G(1-2(a+n))} \tag{8}$$

expressed through partition functions (of deformed) $SU(2)$ gauge theory.

- $t \sim \Lambda^4$, (a, η) are two yet *independent* integration constants, t^{a^2} – classical part ;
- Barnes G-functions $G(a+1) = \Gamma(a)G(a) \underset{a \to \infty}{\sim} \exp\left(\frac{1}{2}a^2 \log a\right)$;
- $\mathcal{B}(a, t) = \sum_{\lambda, \mu} t^{|\lambda|+|\mu|} \left(\frac{a+\dots}{a+\dots}\right)$: Nekrasov instanton partition function, $|\lambda| + |\mu| = k$;

Painlevé/SYM

- Analytic properties of the Painlevé solutions contain important information about non-perturbative SYM: Already $t \sim \Lambda^4$ gives $4 = 2N$ pure $SU(2)$ beta-function ...
- Expansion in $t = \Lambda^4$ at $t \to 0$ and in $t^{-1/4} = \Lambda^{-1}$ at $t \to \infty$;
- Non-autonomous Toda equation

$$\partial_t t \partial_t \log \mathcal{T}(t) = -t^{1/2} \frac{\mathcal{T}_1(t)^2}{\mathcal{T}(t)^2} \tag{9}$$

an analog of $\frac{\partial^2 \mathcal{F}}{\partial \tau^2} = \exp \frac{\partial^2 \mathcal{F}}{\partial a^2}$.
- ‘Gravitational flows’: the Nakajima-Yoshioka *blow-up equations* from simple analysis.

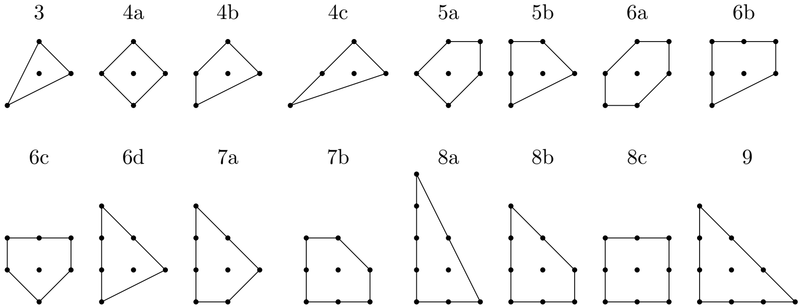
Quivers, gauge theories and Poisson manifolds

- Simple combinatorial objects: vertices with arrows;
- Consistent pictures for gauge theories, generally with bi-fundamentals (Standard Model?);
- More applications: symplectic or Poisson structures
  - Lattices of BPS-charges in supersymmetric theories;
  - Extended phase spaces of integrable systems.

'5d' or relativistic IS on Poisson cluster varieties

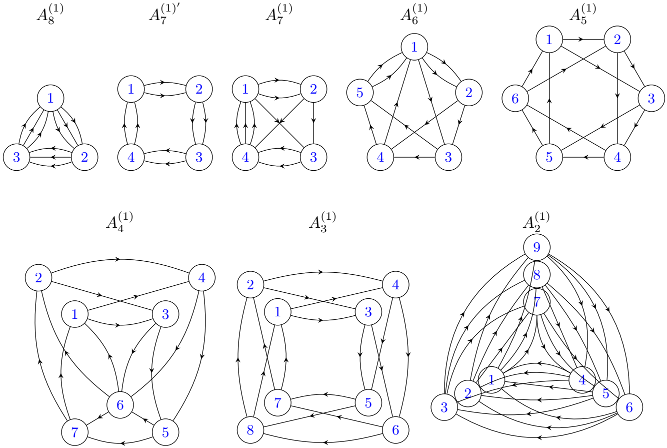
Application: Painlevé Newton Polygons

with a single internal point and $3 \leq B \leq 9$ boundary points:



Here Σ: $f_\Delta(\lambda, \mu) = \sum_{(a,b)\in\Delta} \lambda^a \mu^b f_{a,b} = 0$ is torus with $g = 1$.

Application: Painlevé quivers



Notations: Sakai classification

$$\frac{A_0^{(1)}}{E_8^{(1)}} \to \frac{A_1^{(1)}}{E_7^{(1)}} \to \frac{A_2^{(1)}}{E_6^{(1)}} \to \frac{A_3^{(1)}}{E_5^{(1)}} \to \frac{A_4^{(1)}}{E_4^{(1)}} \to \frac{A_5^{(1)}}{E_3^{(1)}} \to \frac{A_6^{(1)}}{E_2^{(1)}} \begin{array}{l} \nearrow \quad \frac{A_7^{(1)}}{\tilde{A}_1^1} \to \frac{A_8^{(1)}}{E_0^{(1)}} \\ \searrow \quad \frac{A_7^{(1)'}}{E_1^{(1)}} \end{array}$$

by (surface type)/(symmetry group)

- $\mathcal{G}_{\mathcal{Q}} \supset \widehat{W}(E_{\#}^{(1)})$;
- $\widehat{W}(E_0^{(1)}) = \mathbb{Z}/3\mathbb{Z}$;
- From $E_1^{(1)} = A_1^{(1)}$ till $E_5^{(1)} = D_5^{(1)}$ q-Painlevé with well-defined 4d limit (from PIII to PVI);
- Higher $E_7^{(1)}$ and $E_8^{(1)}$ do not have corresponding (naive) $g = 1$ triangles.

Predictions: 5d field theory

- Well-defined solutions exist for $N_f \leq 2N(=4)$): restriction from 4d β-function (IR!);
- q-Painlevé systems themselves exist for $N_f \leq 7$, when ... 5d theory (in UV!) can be defined (Seiberg, 1995);
- $\mathcal{G}_{\mathcal{Q}} \supset \widehat{W} \supset W$ extends to global symmetry of 5d theory in UV (?!);