# Neutrino oscillations in vacuum

# 5 Quantum-mechanical treatment.

## 5.1 Angels & hippopotami.

According to the current theoretical understanding, the neutrino fields/states of definite flavor are superpositions of the fields/states with definite, generally different masses [and vice versa]:

$$\nu_\alpha = \sum_i V_{\alpha i} \nu_i \qquad \text{for neutrino fields,}$$

$$|\nu_\alpha\rangle = \sum_i V^*_{\alpha i} |\nu_i\rangle \qquad \text{for neutrino states;}$$

$$\alpha = e, \mu, \tau, \quad i = 1, 2, 3, \ldots$$

Here $V_{\alpha i}$ are the elements of the Pontecorvo-Maki-Nakagawa-Sakata neutrino vacuum mixing matrix $\mathbf{V}$.

This concept leads to the possibility of transitions between different flavor neutrinos, $\nu_\alpha \longleftrightarrow \nu_\beta$, phenomenon known as neutrino flavor oscillations.

Let us introduce two types of neutrino eigenstates:

- The flavor neutrino eigenstates which can be written as a vector

$$|\boldsymbol{\nu}\rangle_f = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle, \ldots)^T \equiv (|\nu_\alpha\rangle)^T$$

are defined as the states which correspond to the charge leptons $\alpha = e, \mu, \tau$. The correspondence is established through the charged current interactions of active neutrinos and charged leptons.

> Together with the standard $\nu$s, $|\boldsymbol{\nu}\rangle_f$ may include also neutrino states allied with additional heavy charged leptons, as well as the states not associated with charge leptons, like sterile neutrinos, $\nu_s$.

In general, the flavor states have no definite masses. Therefore, they can have either definite momentum, or definite energy but not both.

- The neutrino mass eigenstates

$$|\boldsymbol{\nu}\rangle_m = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle, \ldots)^T \equiv (|\nu_k\rangle)^T$$

are, by definition, the states with the definite masses $m_k$, $k = 1, 2, 3, \ldots$.

Since $|\nu_\alpha\rangle$ and $|\nu_k\rangle$ are not identical, they are related to each other through a unitary transformation

$$|\nu_\alpha\rangle = \sum_k \hat{V}_{\alpha k} |\nu_k\rangle \quad \text{or} \quad |\boldsymbol{\nu}\rangle_f = \hat{\mathbf{V}} |\boldsymbol{\nu}\rangle_m,$$

where $\hat{\mathbf{V}} = \parallel \hat{V}_{\alpha k} \parallel$ is a unitary (in general, $N \times N$) matrix.

To find out the correspondence between $\hat{\mathbf{V}}$ and the PMNS mixing matrix $\mathbf{V}$ we can normalize the "$f$" and "$m$" states by the following conditions

$$\langle 0|\nu_{\alpha L}(x)|\nu_{\alpha'}\rangle = \delta_{\alpha\alpha'} \quad \text{and} \quad \langle 0|\nu_{kL}(x)|\nu_{k'}\rangle = \delta_{kk'}.$$

From these conditions we obtain

$$\sum_k V_{\alpha k}\hat{V}_{\alpha' k} = \delta_{\alpha\alpha'} \quad \text{and} \quad \sum_\alpha V_{\alpha k}\hat{V}_{\alpha k'} = \delta_{kk'}.$$

Therefore

$$\hat{\mathbf{V}} \equiv \mathbf{V}^\dagger$$

and

$$\boxed{|\boldsymbol{\nu}\rangle_f = \mathbf{V}^\dagger|\boldsymbol{\nu}\rangle_m \quad \Longleftrightarrow \quad |\boldsymbol{\nu}\rangle_m = \mathbf{V}|\boldsymbol{\nu}\rangle_f.} \tag{11}$$

The time evolution of a single mass eigenstate $|\nu_k\rangle$ with momentum $p_\nu$ is trivial,

$$i\frac{d}{dt}|\nu_k(t)\rangle = E_k|\nu_k(t)\rangle \quad \Longrightarrow \quad |\nu_k(t)\rangle = e^{-iE_k(t-t_0)}|\nu_k(t_0)\rangle,$$

where $E_k = \sqrt{p_\nu^2 + m_k^2}$ is the total energy in the state $|\nu_k\rangle$. Now, assuming that all $N$ states $|\nu_k\rangle$ have the same momentum, one can write

$$i\frac{d}{dt}|\boldsymbol{\nu}(t)\rangle_m = \mathbf{H}_0|\boldsymbol{\nu}(t)\rangle_m, \quad \text{where} \quad \mathbf{H}_0 = \text{diag}\,(E_1, E_2, E_3, \ldots). \tag{12}$$

From Eqs. (11) and (12) we have

$$i\frac{d}{dt}|\boldsymbol{\nu}(t)\rangle_f = \mathbf{V}^\dagger \mathbf{H}_0 \mathbf{V} |\boldsymbol{\nu}(t)\rangle_f. \tag{13}$$

Solution to this equation is obvious:

$$\begin{aligned} |\boldsymbol{\nu}(t)\rangle_f &= \mathbf{V}^\dagger e^{-i\mathbf{H}_0(t-t_0)} \mathbf{V} \, |\boldsymbol{\nu}(t_0)\rangle_f \\ &= \mathbf{V}^\dagger \, \mathsf{diag}\left(e^{-iE_1(t-t_0)}, e^{-iE_2(t-t_0)}, \ldots\right) \, \mathbf{V} \, |\boldsymbol{\nu}(t_0)\rangle_f. \end{aligned} \tag{14}$$

Now we can derive the survival and transition probabilities

$$\begin{aligned} P_{\alpha\beta}(t-t_0) = P\left[\nu_\alpha(t_0) \to \nu_\beta(t)\right] = & \quad |\langle \nu_\beta(t)|\nu_\alpha(t_0)\rangle|^2 \\ = \Big| \sum_k V_{\alpha k} V^*_{\beta k} \, \exp\left[iE_k(t-t_0)\right] \Big|^2 & \\ = \sum_{jk} V_{\alpha j} V_{\beta k} \left(V_{\alpha k} V_{\beta j}\right)^* \, \exp\left[i(E_j - E_k)(t-t_0)\right]. & \end{aligned}$$

In the ultrarelativistic limit $p_\nu^2 \gg m_k^2$, which is undoubtedly valid for all interesting circumstances (except relic neutrinos),

$$E_k = \sqrt{p_\nu^2 + m_k^2} \approx p_\nu + \frac{m_k^2}{2p_\nu} \approx E_\nu + \frac{m_k^2}{2E_\nu}.$$

Therefore in very good approximation

$$P_{\alpha\beta}(t-t_0)=\sum_{jk}V_{\alpha j}V_{\beta k}\left(V_{\alpha k}V_{\beta j}\right)^*\,\exp\left[\frac{i\Delta m_{jk}^2(t-t_0)}{2E_\nu}\right].$$

As a rule, there is no way to measure $t_0$ and $t$ in the same experiment.[a] But it is usually possible to measure the distance $L$ between the source and detector. So we have to connect $t-t_0$ with $L$. It is easy to do in the standard ultrarelativistic approximation,

$$v_k=\frac{p_\nu}{E_k}\simeq 1-\frac{m_k^2}{2E_\nu^2}=1-0.5\times 10^{-14}\left(\frac{m_k}{0.1\text{ eV}}\right)^2\left(\frac{1\text{ MeV}}{E_\nu}\right)^2\simeq 1,$$

from which it almost evidently follows that $t-t_0\approx L$. Finally we arrive at the following formula

$$P_{\alpha\beta}(L)=\sum_{jk}V_{\alpha j}V_{\beta k}\left(V_{\alpha k}V_{\beta j}\right)^*\,\exp\left(\frac{2i\pi L}{L_{jk}}\right),\quad L_{jk}=\frac{4\pi E_\nu}{\Delta m_{jk}^2},\tag{15}$$

where $L_{jk}$ (or more exactly $|L_{jk}|=|L_{kj}|$) are the so-called neutrino oscillation lengths.

It is straightforward to prove that the QM formula satisfies the probability conservation law:

$$\sum_\alpha P_{\alpha\beta}(L)=\sum_\beta P_{\alpha\beta}(L)=1.$$

The range of applicability of the standard quantum-mechanical approach is limited but enough for the interpretation of essentially all modern experiments with accelerator, reactor, atmospheric, solar, and astrophysical neutrino beams.

[a] Important exceptions will be discussed in the special section.

## 5.2 Energy conservation.

Although the energy of the state with definite flavor, $|\nu_\alpha(L)\rangle = |\nu_\alpha(t)\rangle$, is not defined, its mean energy, $\langle E_\alpha(t)\rangle = \langle \nu_\alpha(t)|\hat{H}|\nu_\alpha(t)\rangle$, is a well-defined and conserved quantity. Indeed,

$$\langle E_\alpha(t)\rangle = \sum_{ij} V_{\alpha i}V^*_{\alpha j}\langle \nu_i(p)|\hat{H}|\nu_j(p)\rangle = \sum_{ij} V_{\alpha i}V^*_{\alpha j}\langle \nu_i(p)|E_i|\nu_j(p)\rangle \equiv \langle E_\alpha\rangle = \text{inv.}$$

$$\langle E_\alpha\rangle = \sum_i |V_{\alpha i}|^2 E_i \simeq p + \sum_i |V_{\alpha i}|^2 \frac{m_i^2}{2p}, \quad \Longrightarrow \quad \sum_\alpha \langle E_\alpha\rangle = \sum_i E_i \simeq 3\left(p + \sum_i \frac{m_i^2}{2p}\right).$$

Moreover, the mean energy of an arbitrary entangled state characterized by a certain density matrix $\boldsymbol{\rho}(t)$ is also conserved. Indeed, let the initial state have the form

$$\boldsymbol{\rho}(0) = \sum_\alpha w_\alpha |\nu_\alpha(0)\rangle\langle \nu_\alpha(0)|,$$

The mean energy of the mixed state at arbitrary time $t$ is then written as

$$\begin{aligned}\langle E(t)\rangle &= \text{Tr}\left(\hat{H}\boldsymbol{\rho}(t)\right) = \text{Tr}\left(\hat{H}e^{-i\hat{H}t}\boldsymbol{\rho}(0)e^{i\hat{H}t}\right) \\ &= \sum_\alpha w_\alpha \sum_{ij} V^*_{\alpha i}V_{\alpha j}e^{-i(E_i-E_j)t}E_i\ \text{Tr}|\nu_i(p)\rangle\langle \nu_j(p)| \\ &= \sum_\alpha w_\alpha \sum_i |V_{\alpha i}|^2 E_i = \text{inv}, \quad \Longrightarrow \quad \langle E(t)\rangle = \sum_\alpha w_\alpha \langle E_\alpha\rangle.\end{aligned}$$

Naturally, $\langle E(t)\rangle = \langle E_\alpha\rangle$ for the pure initial state $|\nu_\alpha(0)\rangle$ (when $\boldsymbol{\rho}(0) = |\nu_\alpha(0)\rangle\langle \nu_\alpha(0)|$).

## 5.3 Simplest example: two-flavor oscillations.

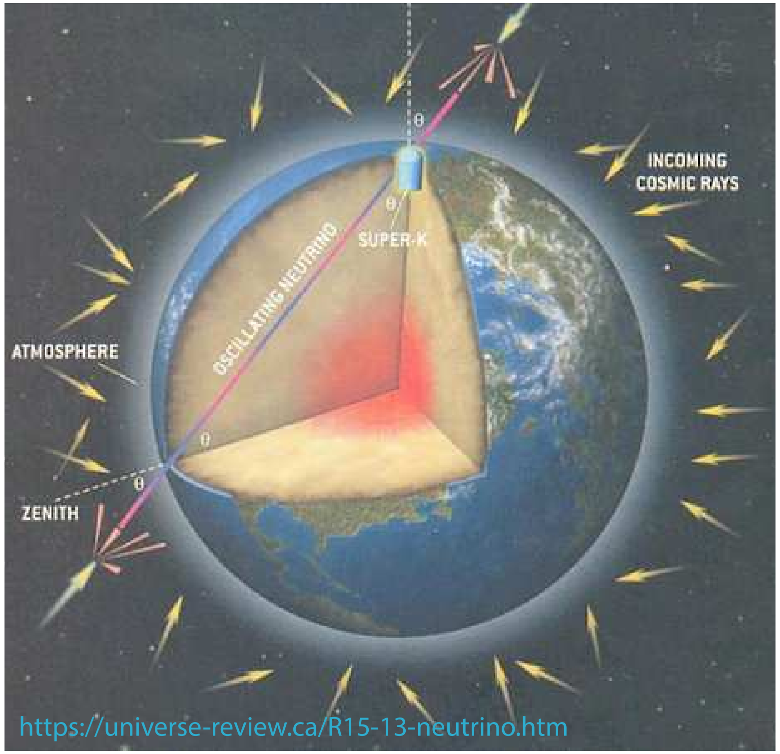
Let's now consider the simplest (toy) 2-flavor model, e.g., with $i = 2, 3$ and $\alpha = \mu, \tau$ (the most favorable due to the SK and other underground experiments). The $2 \times 2$ vacuum mixing matrix can be parametrized (due to the unitarity) with a single parameter, $\theta$ $(= \theta_{23})$, the vacuum mixing angle,

$$\mathbf{V} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad 0 \le \theta \le \frac{\pi}{2}.$$

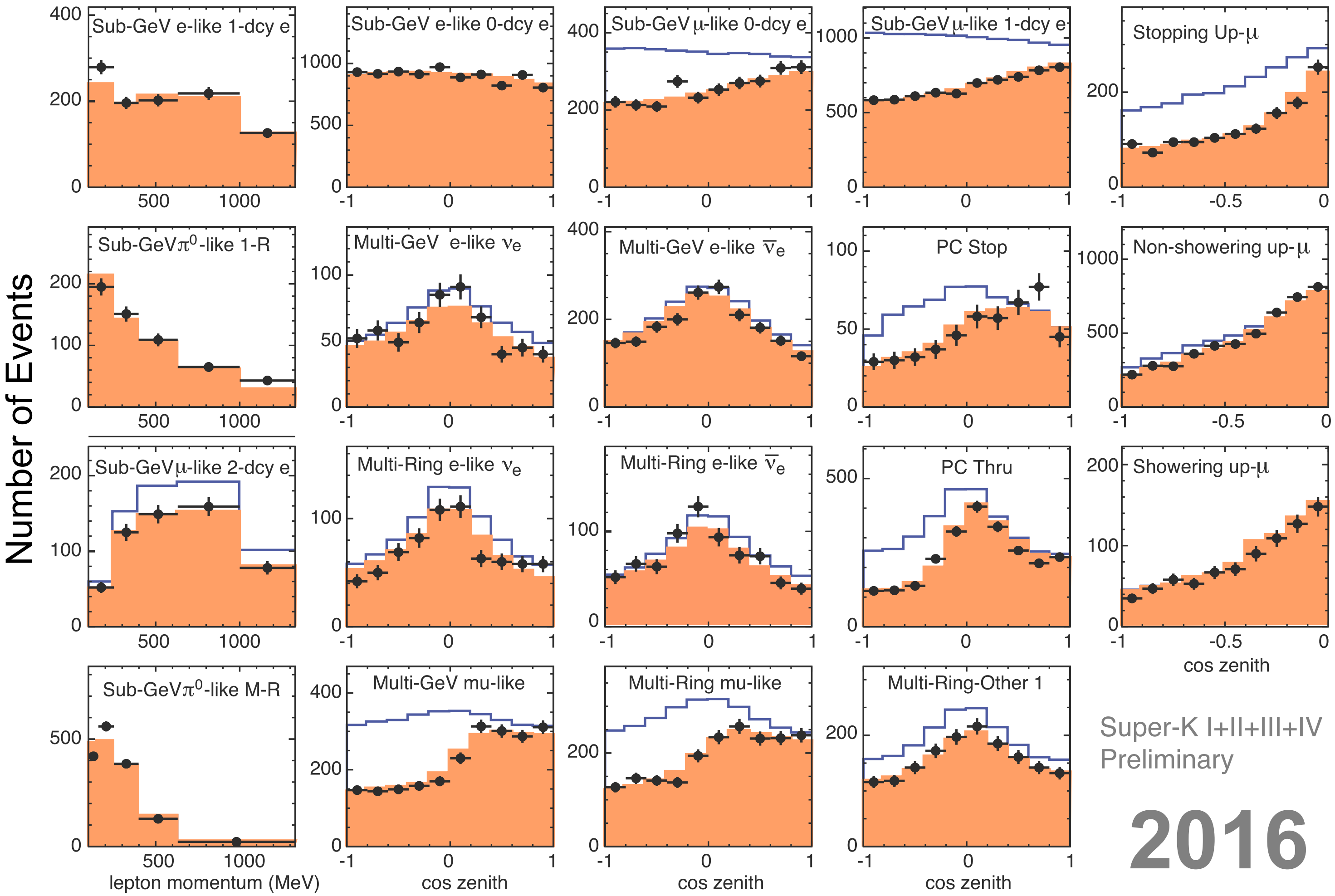
In this model, Eq. (15) then becomes very simple and transparent:

$$P_{\mu\tau}(L) = P_{\tau\mu}(L) = \frac{1}{2}\sin^2 2\theta \left[1 - \cos\left(\frac{2\pi L}{L_\nu}\right)\right],$$

$$L_\nu \equiv L_{23} = \frac{4\pi E_\nu}{\Delta m^2_{23}} \approx 2R_\oplus \left(\frac{E_\nu}{10\ \text{GeV}}\right)\left(\frac{0.002\ \text{eV}^2}{\Delta m^2_{23}}\right).$$

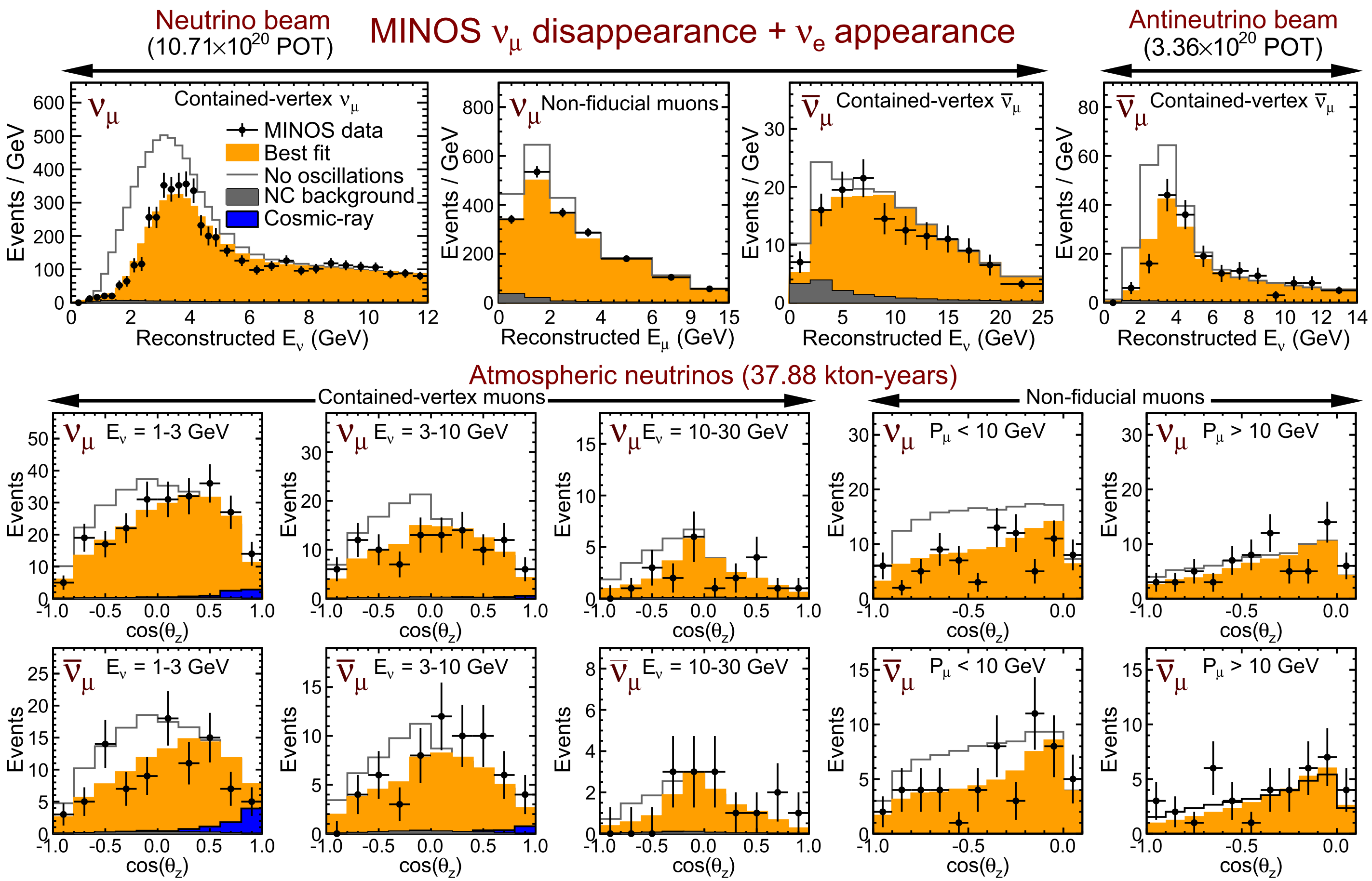



Here $R_\oplus$ is the mean radius of Earth and 10 GeV is a typical energy in the (very wide) atmospheric neutrino spectrum.
Since Earth provides variable "baseline" [from about 15 km to about 12700 km], it is surprisingly suitable for studying the atmospheric (as well as accelerator and reactor) neutrino oscillations in rather wide range of the oscillation parameters.




Zenith angle and momentum distributions for atmospheric neutrino subsamples used for an analyses by Super-Kamiokande to study subleading effects, preferences for mass hierarchy and $\delta_{\mathrm{CP}}$, as well as searches for astrophysical sources such as dark matter annihilation.

[From T. Kajita *et al.* (for the Super-Kamiokande Collaboration), "Establishing atmospheric neutrino oscillations with Super-Kamiokande, "Nucl. Phys. B **908** (2016) 14–29.]




The event spectra at MINOS from $10.71 \times 10^{20}$ POT FHC ($\nu_\mu$-dominated) mode, $3.36 \times 10^{20}$ POT RHC ($\overline{\nu}_\mu$-dominated) mode and $37.88$ kt·yrs of atmospheric data. The data are shown compared to the prediction in absence of oscillations (grey lines) and to the best-fit prediction (red). The beam histograms (top) also include the NC background component (filled grey) and the atmospheric histograms (bottom) include the cosmic-ray background contribution filled blue).

[From L. H. Whitehead (For the MINOS Collaboration), "Neutrino oscillations with MINOS and MINOS+," Nucl. Phys. B **908** (2016) 130–150.]

## 5.4 Summary of the standard QM theory.

The standard assumptions are intuitively transparent and (almost) commonly accepted.

[1] The neutrino flavor states $|\nu_\alpha\rangle$ associated with the charged leptons $\alpha = e, \mu, \tau$ (that is having definite lepton numbers) are not identical to the neutrino mass eigenstates $|\nu_i\rangle$ with the definite masses $m_i$ $(i = 1, 2, 3)$.

Both sets of states are orthonormal: $\langle\nu_\beta|\nu_\alpha\rangle = \delta_{\alpha\beta}$, $\langle\nu_j|\nu_i\rangle = \delta_{ij}$.

$$\Downarrow$$

They are related to each other through a unitary transformation $\mathbf{V} = ||V_{\alpha i}||$, $\mathbf{V}\mathbf{V}^\dagger = 1$,

$$|\nu_\alpha\rangle = \sum_i V^*_{\alpha i}|\nu_i\rangle, \quad |\nu_i\rangle = \sum_\alpha V_{\alpha i}|\nu_\alpha\rangle.$$

[2] Massive neutrino states originated from any reaction or decay have the same definite momenta $\mathbf{p}_\nu$ ["equal momentum (EM) assumption"]. [a]

| To simplify matter, we do not consider exotic processes with multiple neutrino production. |
|---|

$$\Downarrow$$

The flavor states $|\nu_\alpha\rangle$ have the same momentum $\mathbf{p}_\nu$ but have no definite mass and energy.

[a]Sometimes – the same definite energies ["equal energy (EE) assumption"].

[3] Neutrino masses are so small that in essentially all experimental circumstances (or, more precisely, in a wide class of reference frames) the neutrinos are ultrarelativistic. Hence

$$E_k = \sqrt{\mathbf{p}_\nu^2 + m_k^2} \simeq |\mathbf{p}_\nu| + \frac{m_k^2}{2|\mathbf{p}_\nu|}.$$

[4] Moreover, in the evolution equation, one can safely replace the time parameter $t$ by the distance $L$ between the neutrino source and detector. [Let's remind that $\hbar = c = 1$.]

The enumerated assumptions are sufficient to derive the nice and commonly accepted expression for the neutrino flavor transition probability [$L_{jk}$ are the neutrino oscillation lengths]:

$$\begin{aligned}
P(\nu_\alpha \to \nu_\beta; L) \equiv P_{\alpha\beta}(L) &= \sum_{jk} V_{\alpha j} V_{\beta k} \left(V_{\alpha k} V_{\beta j}\right)^* \exp\left(\frac{2i\pi L}{L_{jk}}\right) \\
&= \sum_j |V_{\alpha j}|^2 |V_{\beta j}|^2 + 2 \sum_{j>k} \left[\mathrm{Re}\left(V_{\alpha j}^* V_{\beta j} V_{\alpha k} V_{\beta k}^*\right) \cos\left(\frac{2\pi L}{L_{jk}}\right)\right. \\
&\qquad \left. + \,\mathrm{Im}\left(V_{\alpha j}^* V_{\beta j} V_{\alpha k} V_{\beta k}^*\right) \sin\left(\frac{2\pi L}{L_{jk}}\right)\right],
\end{aligned}$$

$$L_{jk} = \frac{4\pi E_\nu}{\Delta m_{jk}^2}, \qquad E_\nu = |\mathbf{p}_\nu|, \qquad \Delta m_{jk}^2 = m_j^2 - m_k^2.$$

Just this result is the basis for the "oscillation interpretation" of the current experiments with the natural and artificial neutrino and antineutrino beams.

## 5.5 Some challenges against the QM approach.

Equal-momentum assumption

Massive neutrinos $\nu_i$ have, by assumption, equal momenta: $\mathbf{p}_i = \mathbf{p}_\nu$ $(i = 1, 2, 3)$.

This key assumption seems to be unphysical being reference-frame (RF) dependent;

if it is true in a certain RF then it is false in another RF moving with the velocity $\mathbf{v}$:

$$E_i' = \Gamma_{\mathbf{v}}\left[E_i - (\mathbf{v}\mathbf{p}_\nu)\right], \quad \mathbf{p}_i' = \mathbf{p}_\nu + \Gamma_{\mathbf{v}}\left[\frac{\Gamma_{\mathbf{v}}(\mathbf{v}\mathbf{p}_\nu)}{\Gamma_{\mathbf{v}}+1} - E_i\right]\mathbf{v},$$

$\Downarrow$ [assuming, as necessary for oscillations, that $m_i \neq m_j$] $\Downarrow$

$$\mathbf{p}_i' - \mathbf{p}_j' = \left(E_j' - E_i'\right)\mathbf{v} = \Gamma_{\mathbf{v}}\left(E_j - E_i\right)\mathbf{v} \neq 0.$$

Treating the Lorentz transformation as active, we conclude that the EM assumption cannot be applied to the non-monoenergetic $\nu$ beams (the case in real-life experiments).

$*$ A similar objection exists against the alternative equal-energy assumption; in that case

$$E_i' - E_j' = \Gamma_{\mathbf{v}}\left(\mathbf{p}_j - \mathbf{p}_i\right)\mathbf{v} \neq 0, \quad \left|\mathbf{p}_i' - \mathbf{p}_j'\right| = \sqrt{\left|\mathbf{p}_i - \mathbf{p}_j\right|^2 + \Gamma_{\mathbf{v}}^2\left[\left(\mathbf{p}_i - \mathbf{p}_j\right)\mathbf{v}\right]^2} \neq 0.$$

$*$ Can the EM (or EE) assumption be at least a good approximation? Alas, no, it cannot.

> Let $\nu_\mu$s arise from $\pi_{\mu 2}$ decays. If the pion beam has a wide momentum spectrum – from subrelativistic to ultrarelativistic (as it is, e.g., for cosmic-ray particles), the EM (or EE) condition cannot be valid even approximately within the whole spectral range of the pion neutrinos.

## Light-ray approximation

The propagation time $T$ is, by assumption, equal to the distance $L$ traveled by the neutrino between production and detection points. But, if the massive neutrino components have the same momentum $\mathbf{p}_\nu$, their velocities are in fact different:

$$\mathbf{v}_i = \frac{\mathbf{p}_\nu}{\sqrt{\mathbf{p}_\nu^2 + m_i^2}} \quad \Longrightarrow \quad |\mathbf{v}_i - \mathbf{v}_j| \approx \frac{\Delta m_{ji}^2}{2E_\nu^2}.$$

One may naively expect that during the time $T$ the neutrino $\nu_i$ travels the distance $L_i = |\mathbf{v}_i|T$; therefore, there must be a spread in distances of each neutrino pair

$$\delta L_{ij} = L_i - L_j \approx \frac{\Delta m_{ji}^2}{2E_\nu^2} L, \quad \text{where} \quad L = cT = T.$$

| $\Delta m_{ji}^2$ | $E_\nu$ | $L$ | $L_{ij}$ | $\lvert\delta L_{ij}\rvert$ |
|---|---|---|---|---|
| $\Delta m_{23}^2$ | 1 GeV | $2R_\oplus$ | $0.1R_\oplus$ | $\sim 10^{-12}$ cm |
| $\Delta m_{23}^2$ | 1 TeV | $R_G \sim 100$ kps | $100R_\oplus$ | $\sim 10^{-4}$ cm |
| $\Delta m_{21}^2$ | 1 MeV | 1 AU | $0.25R_\oplus$ | $\sim 10^{-3}$ cm |

The values of $\delta L_{ij}$ listed in the Table seem to be **fantastically** small. But

Are they sufficiently small to preserve the coherence in any circumstance?

In other words:

What is the natural scale of the distances and times?

Can light neutrinos oscillate into heavy ones or vise versa?
[Can active neutrinos oscillate into sterile ones or vise versa?]

The naive QM answer is Yes. Why not? If, at least, both $\nu_\alpha$ (light) and $\nu_s$ (heavy) are ultrarelativistic [ $|\mathbf{p}_\nu| \gg \max(m_1, m_2, m_3, \ldots, M)$, ] one obtains the same formula for the oscillation probability $P_{\alpha s}(L)$, since the QM formalism has no any limitation to the neutrino mass hierarchy.

Possibility of such transitions is a basis for many speculations in astrophysics and cosmology.

But! Assume again that the neutrino source is $\pi_{\mu 2}$ decay and $M > m_\pi$. Then the transition $\nu_\alpha \to \nu_s$ in the pion rest frame is forbidden by the energy conservation.

$$\Downarrow$$

There must be some limitations & flaws in the QM formula. What are they?

Do relic neutrinos oscillate?

Most likely the lightest relic neutrinos are always relativistic or even ultrarelativistic, while heavier ones become subrelativistic and then non-relativistic as the universe expands.
The naive QM approach does not know how to handle such a set of neutrinos.

Does the motion of the neutrino source affect the transition probabilities?

To answer these and similar questions

one has to unload the UR approximation & develop a covariant formalism.

**In the QFT approach:** the effective (most probable) energies and momenta of virtual $\nu_i$s are *found to be* functions of the masses, most probable momenta and momentum spreads of all particles (wave packets) involved into the neutrino production and detection processes.
In particular, in the two limiting cases – ultrarelativistic (UR) and nonrelativistic (NR):

**Ultrarelativistic case**

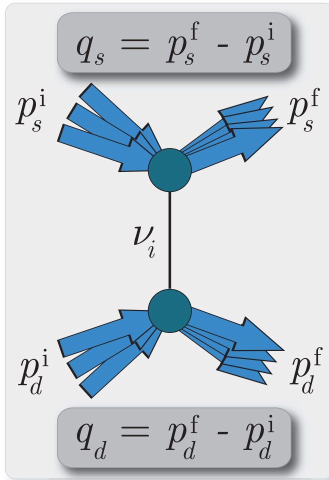
$(|q^0_{s,d}| \sim |\mathbf{q}_{s,d}| \gg m_i)$

$$\begin{cases} E_i = E_\nu \left[1 - \mathfrak{n} r_i - \mathfrak{m} r_i^2 + \ldots\right], \\ |\mathbf{p}_i| = E_\nu \left[1 - (\mathfrak{n}+1)\, r_i - \left(\mathfrak{m} + \mathfrak{n} + \frac{1}{2}\right) r_i^2 + \ldots\right], \\ v_i = 1 - r_i - \left(2\mathfrak{n} + \frac{1}{2}\right) r_i^2 + \ldots < 1, \end{cases}$$

**Nonrelativistic case**

$(|q^0_{s,d}| \sim m_i \gg |\mathbf{q}_{s,d}|)$

$$\begin{cases} E_i = m_i + \frac{m_i v_i^2}{2}\left(1 + \frac{3}{4}\delta_i + \ldots\right), \\ |\mathbf{p}_i| = m_i v_i \left(1 + \frac{1}{2}\delta_i + \ldots\right), \\ v_i \approx \frac{\boldsymbol{\varrho}_i l}{1 + \varrho_i^0} \ll 1, \end{cases}$$




$$E_\nu \approx q_s^0 \approx -q_d^0, \quad r_i = \frac{m_i^2}{2E_\nu^2} \ll 1 \text{ (UR)},$$

$$\varrho_i^\mu = \frac{1}{m_i \mathscr{R}}\left[\widetilde{\mathfrak{R}}_s^{\mu 0}\left(m_i - q_s^0\right) + \widetilde{\mathfrak{R}}_d^{\mu 0}\left(m_i + q_d^0\right) - \widetilde{\mathfrak{R}}_s^{\mu k} q_s^k + \widetilde{\mathfrak{R}}_d^{\mu k} q_d^k\right], \quad |\varrho_i^\mu| \ll 1 \text{ (NR)}.$$

## Definite momentum assumption

In the naive QM approach, the assumed definite momenta of neutrinos (both $\nu_\alpha$ and $\nu_i$) imply that the spatial coordinates of neutrino production ($\mathbf{X}_s$) and detection ($\mathbf{X}_d$) are fully uncertain (Heisenberg's principle).

$$\Downarrow$$

The distance $L = |\mathbf{X}_d - \mathbf{X}_s|$ is uncertain too, that makes the standard QM formula for the flavor transition probabilities to be strictly speaking senseless.

In the correct theory, the neutrino momentum uncertainty $\delta|\mathbf{p}_\nu|$ must be at least of the order of $\min(1/D_s, 1/D_d)$, where $D_s$ and $D_d$ are the characteristic dimensions of the source and detector "machines" along the neutrino beam.

$$\Downarrow$$

The neutrino states must be some wave packets (WP) [though having very small spreads] dependent, in general, on the quantum states of the particles [or, more exactly, also WPs] which participate in the production and detection processes.

**In the QFT approach:** the effective WPs of virtual UR $\nu_i$s are *found to be*

$$\psi_i^{(*)} = \exp\left\{\pm i(p_i X_{s,d}) - \frac{\widetilde{\mathfrak{D}}_i^2}{E_\nu^2}\left[(p_i X)^2 - m_i^2 X^2\right]\right\}, \qquad X = X_d - X_s,$$

where $p_i = (E_i, \mathbf{p}_i)$ and $X_{s,d}$ are the 4-vectors which characterize the space-time location of the $\nu$ production and detection processes, while $\widetilde{\mathfrak{D}}_i$ are certain (in general, complex-valued) functions of the masses, mean momenta and momentum spreads of all particles involved into these processes. [$\widetilde{\mathfrak{D}}_i/E_\nu$ and thereby $\psi_i$ are Lorentz invariants.]

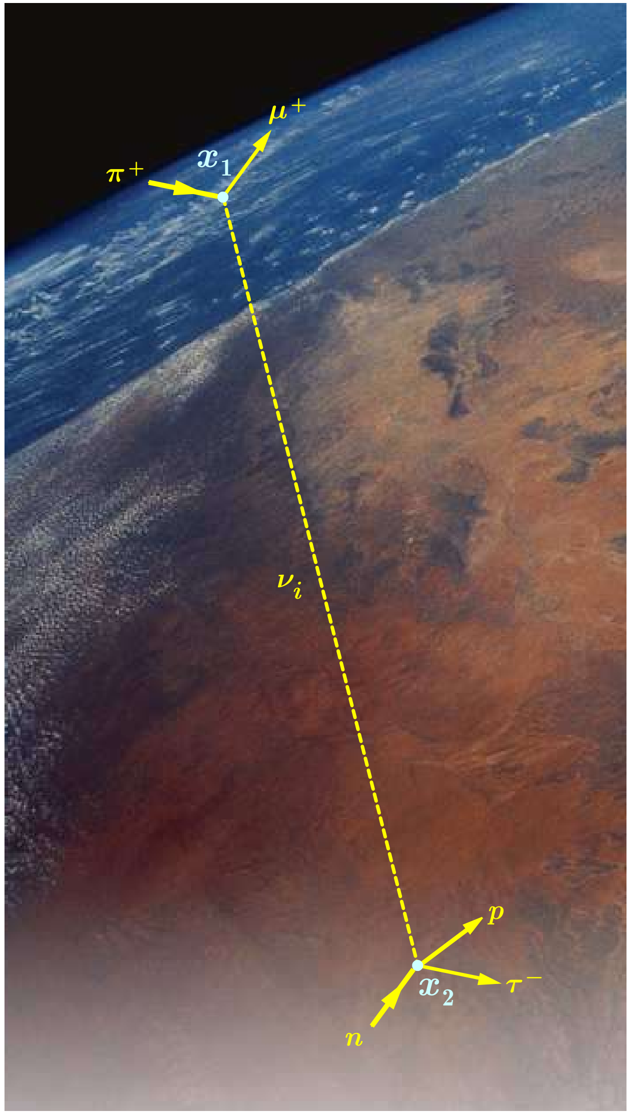
## 5.6 The aims and concepts of the field-theoretical approach.

**The main purposes:**

To define the domain of applicability of the standard quantum-mechanical (QM) theory of vacuum neutrino oscillations and obtain the QFT corrections to it.

**The basic concepts:**

- The "$\nu$-oscillation" phenomenon in QFT is nothing else than a result of interference of the macroscopic Feynman diagrams perturbatively describing the lepton number violating processes with the massive neutrino fields as internal lines (propagators).
- The external lines of the macrodiagrams are wave packets rather than plane waves (therefore the standard $S$ matrix approach should be revised).
- The external wave packet states are the covariant superpositions of the standard one-particle Fock states, satisfying a correspondence principle.

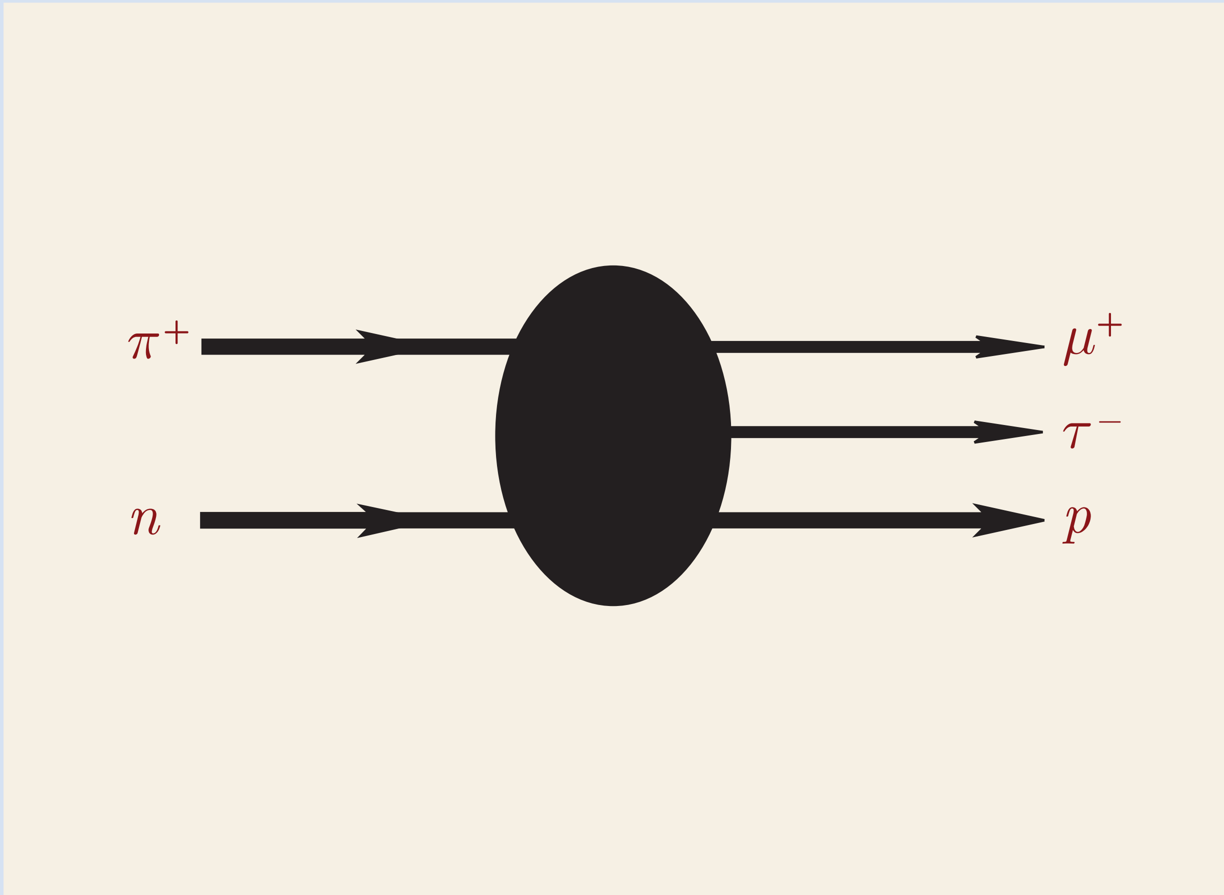



**References:** D. V. Naumov & VN, J. Phys. G **37** (2010) 105014, arXiv:1008.0306 [hep-ph]; Russ. Phys. J. **53** (2010) 549–574; arXiv:1110.0989 [hep-ph]; ЭЧАЯ **51** (2020) 1–209 [Phys. Part. Nucl. **51** (2020) 1–106].

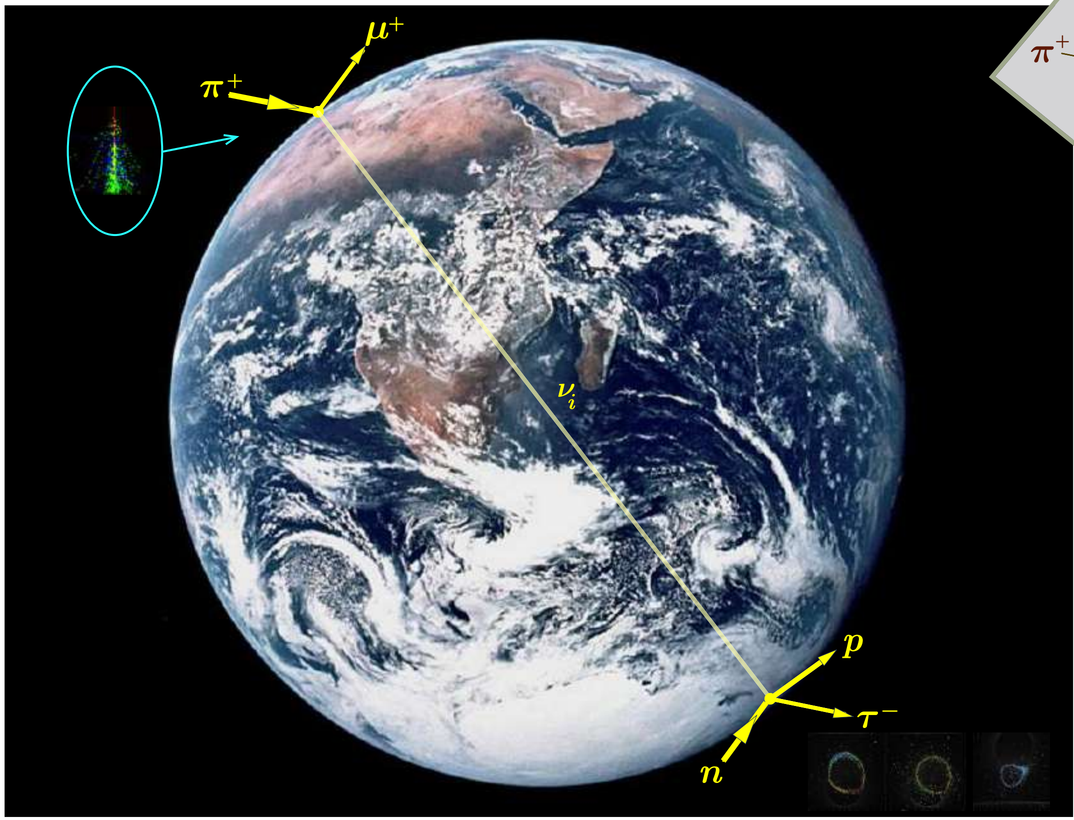
## 5.7 A sketch of the approach.

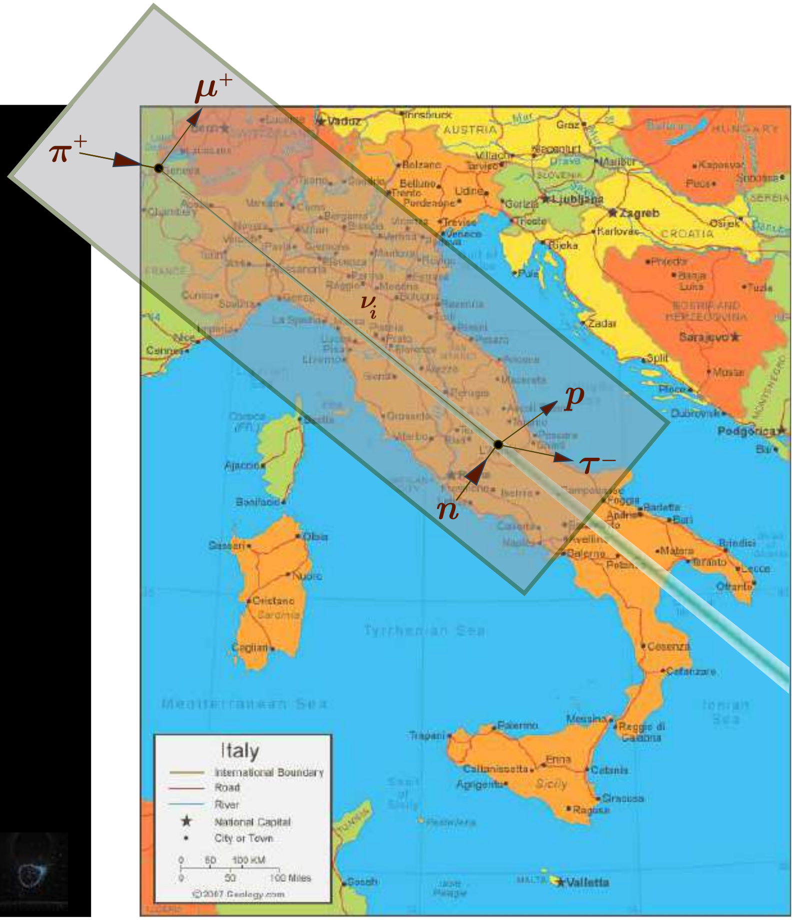
Let us first consider the basics of the QFT approach using the simplest example.

### 5.7.1 QFT approach by the example of the reaction $\pi \oplus n \to \mu \oplus \tau p$.

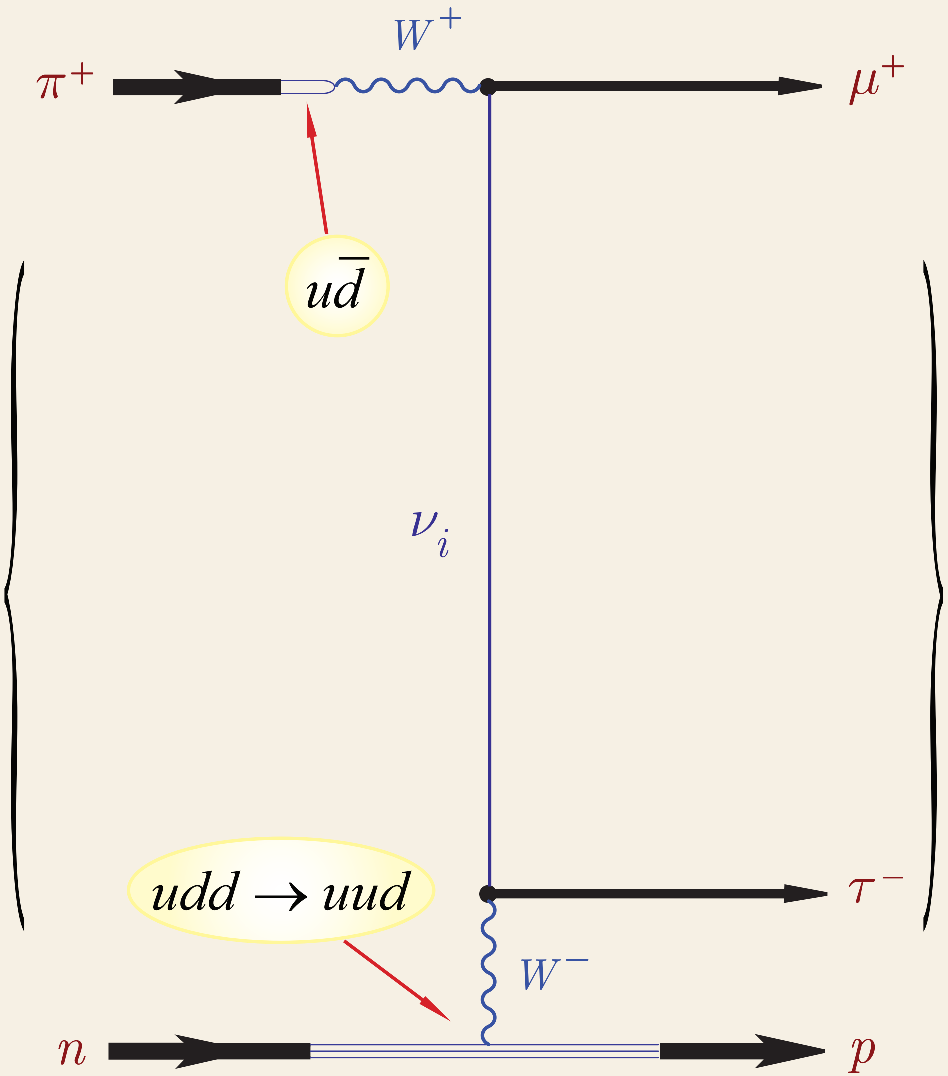


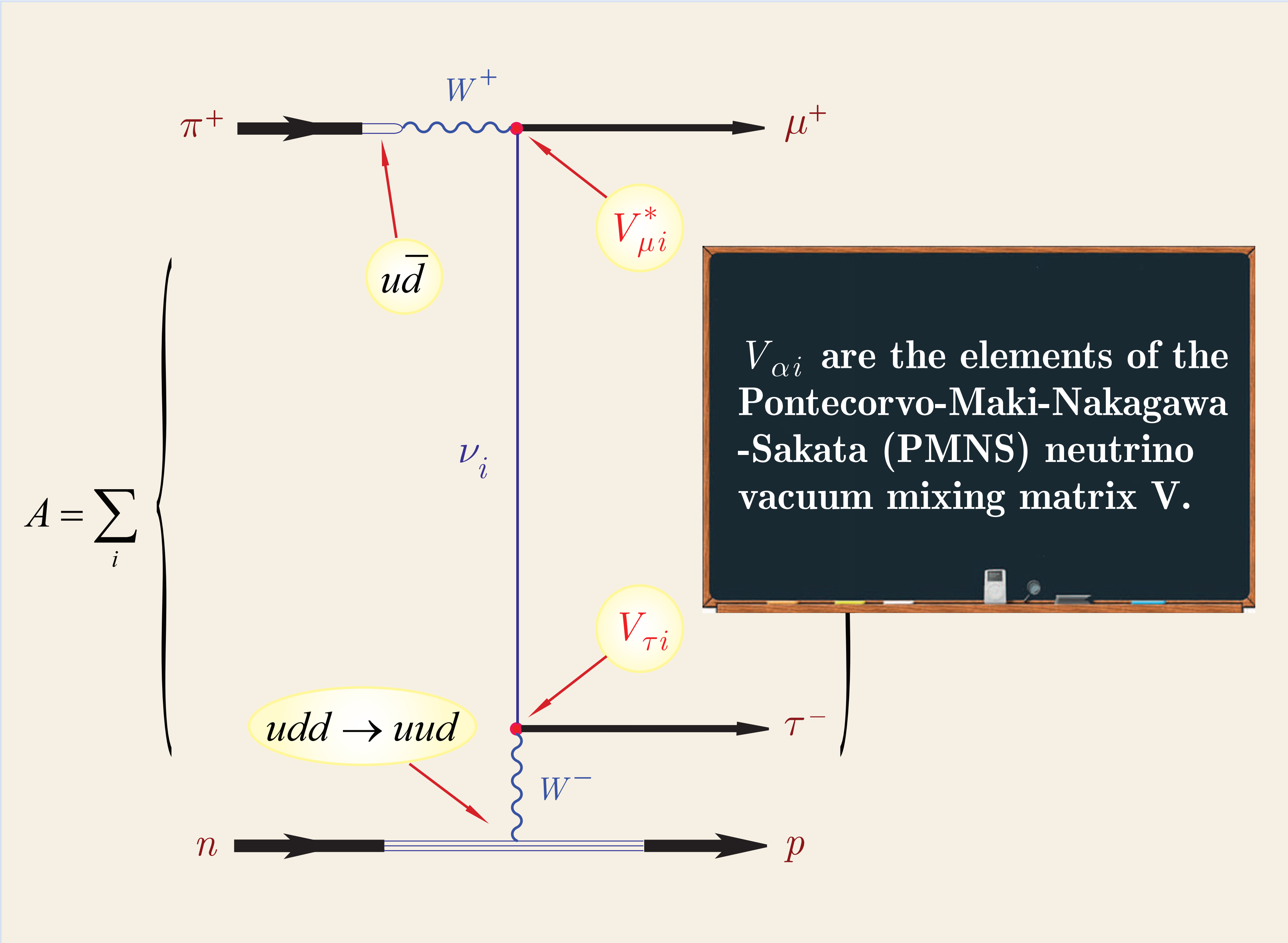
The rare reactions $\pi^+ \oplus n \to \mu^+ \oplus \tau^- p + \ldots$ were (indirectly) detected by several underground experiments (Kamiokande, IMB, Super-Kamiokande) with atmospheric neutrinos. In 2010, OPERA experiment (INFN, LNGS) with the CNGS neutrino beam announced the direct observation of the first $\tau^-$ candidate event; six candidates were recorded in several years of the detector operation.



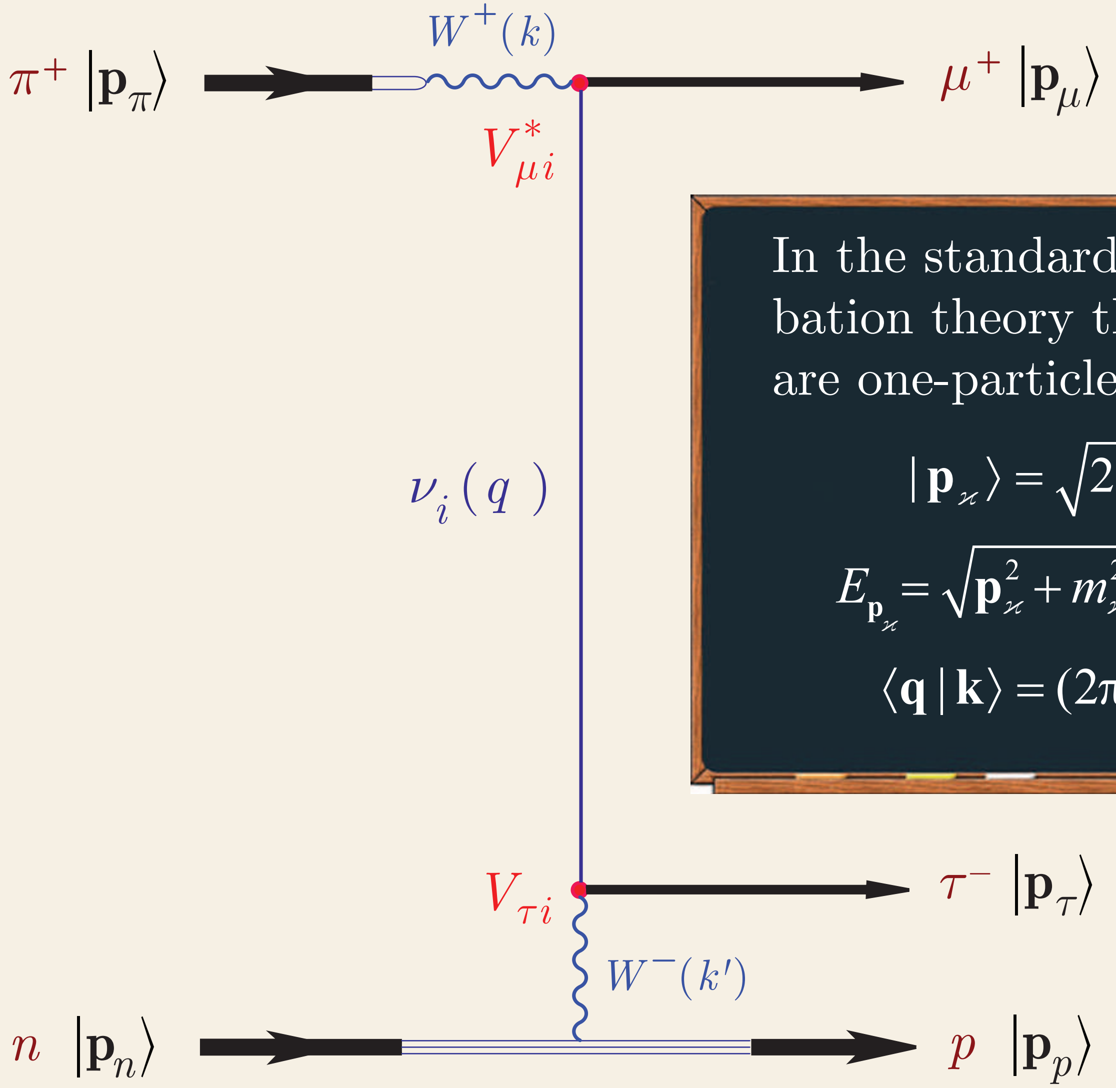



$A = \sum_i$





$W^+$
$\pi^+$
$\mu^+$
$V^*_{\mu i}$
$u\bar{d}$
$V_{\alpha i}$ are the elements of the Pontecorvo-Maki-Nakagawa -Sakata (PMNS) neutrino vacuum mixing matrix V.
$\nu_i$
$A = \sum_i$
$V_{\tau i}$
$udd \to uud$
$\tau^-$
$W^-$
$n$
$p$

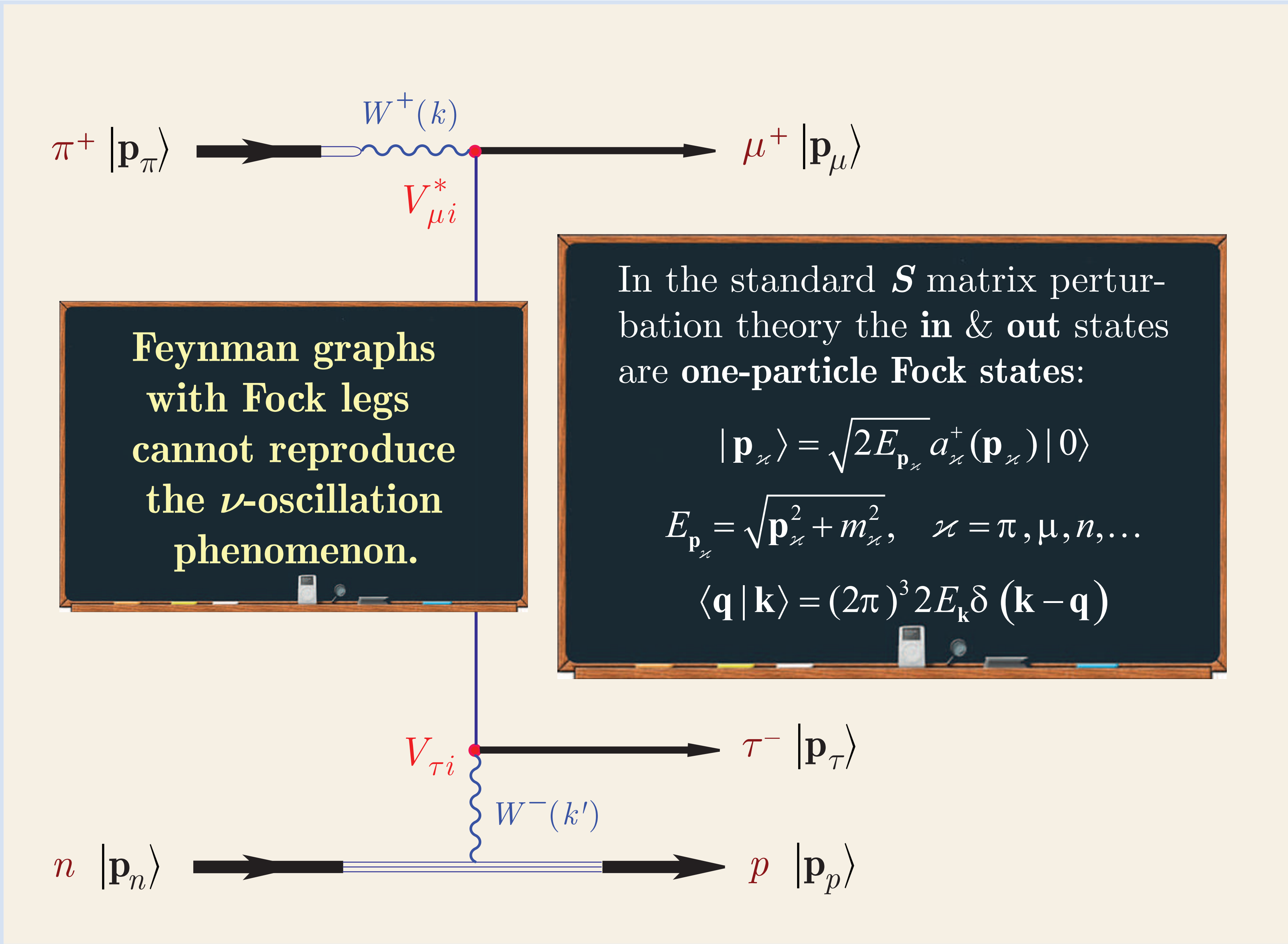


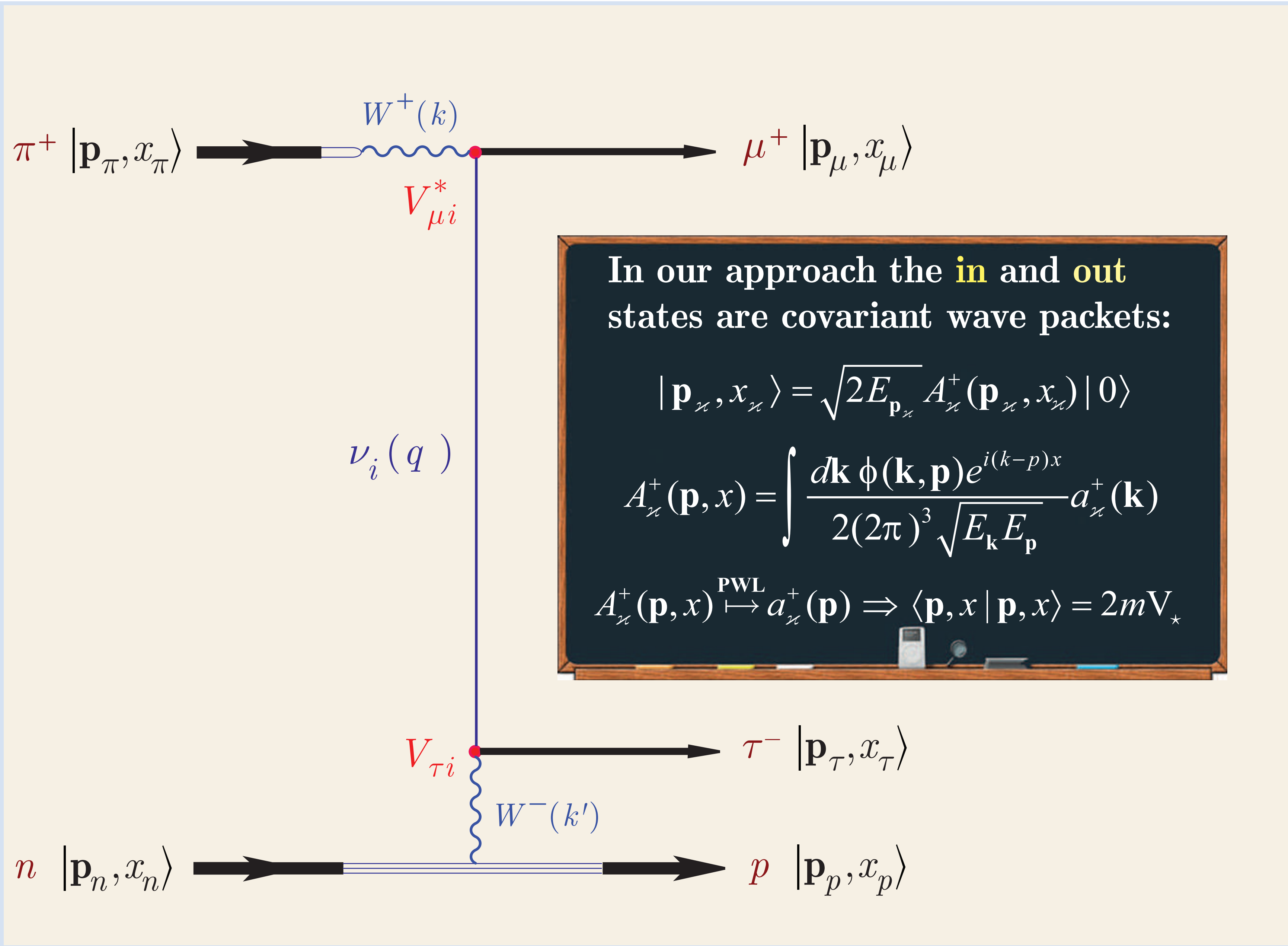

In the standard ***S*** matrix perturbation theory the **in** & **out** states are one-particle **Fock states**:

$$|\mathbf{p}_\varkappa\rangle = \sqrt{2E_{\mathbf{p}_\varkappa}}\, a^+_\varkappa(\mathbf{p}_\varkappa)\,|0\rangle$$

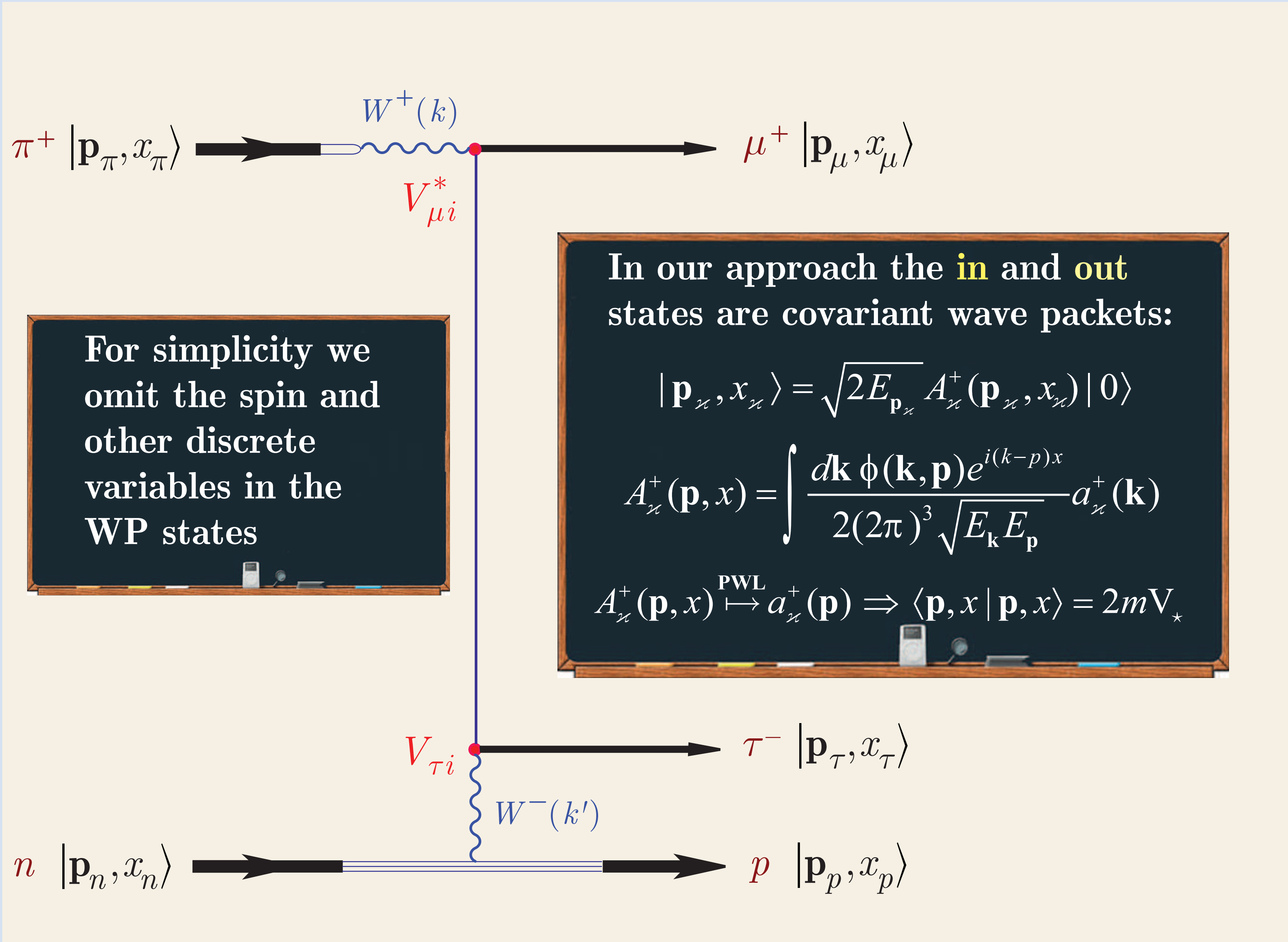
$$E_{\mathbf{p}_\varkappa} = \sqrt{\mathbf{p}^2_\varkappa + m^2_\varkappa}, \quad \varkappa = \pi, \mu, n, \ldots$$

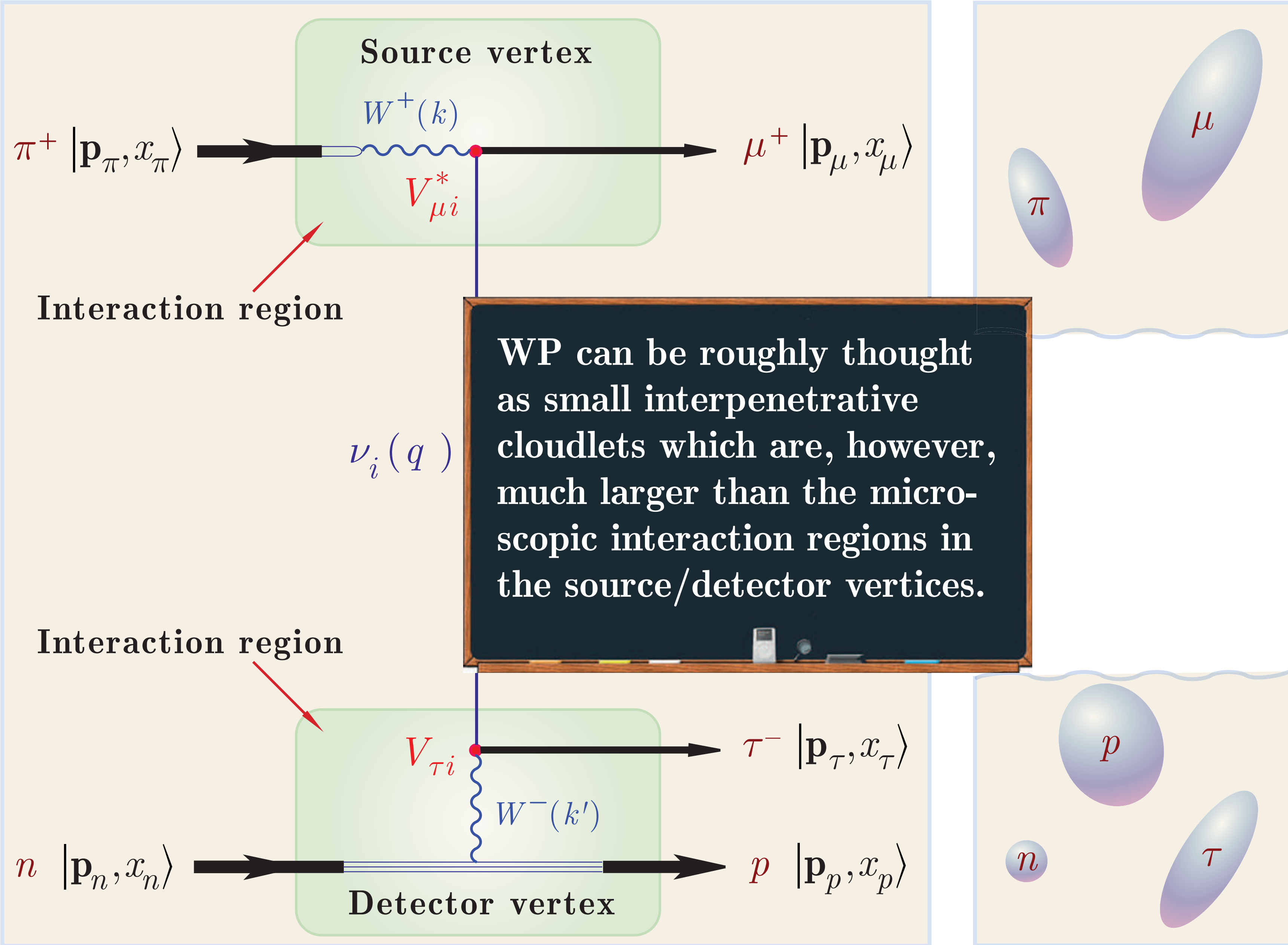
$$\langle \mathbf{q}\,|\,\mathbf{k}\rangle = (2\pi)^3\, 2E_{\mathbf{k}}\,\delta\left(\mathbf{k}-\mathbf{q}\right)$$

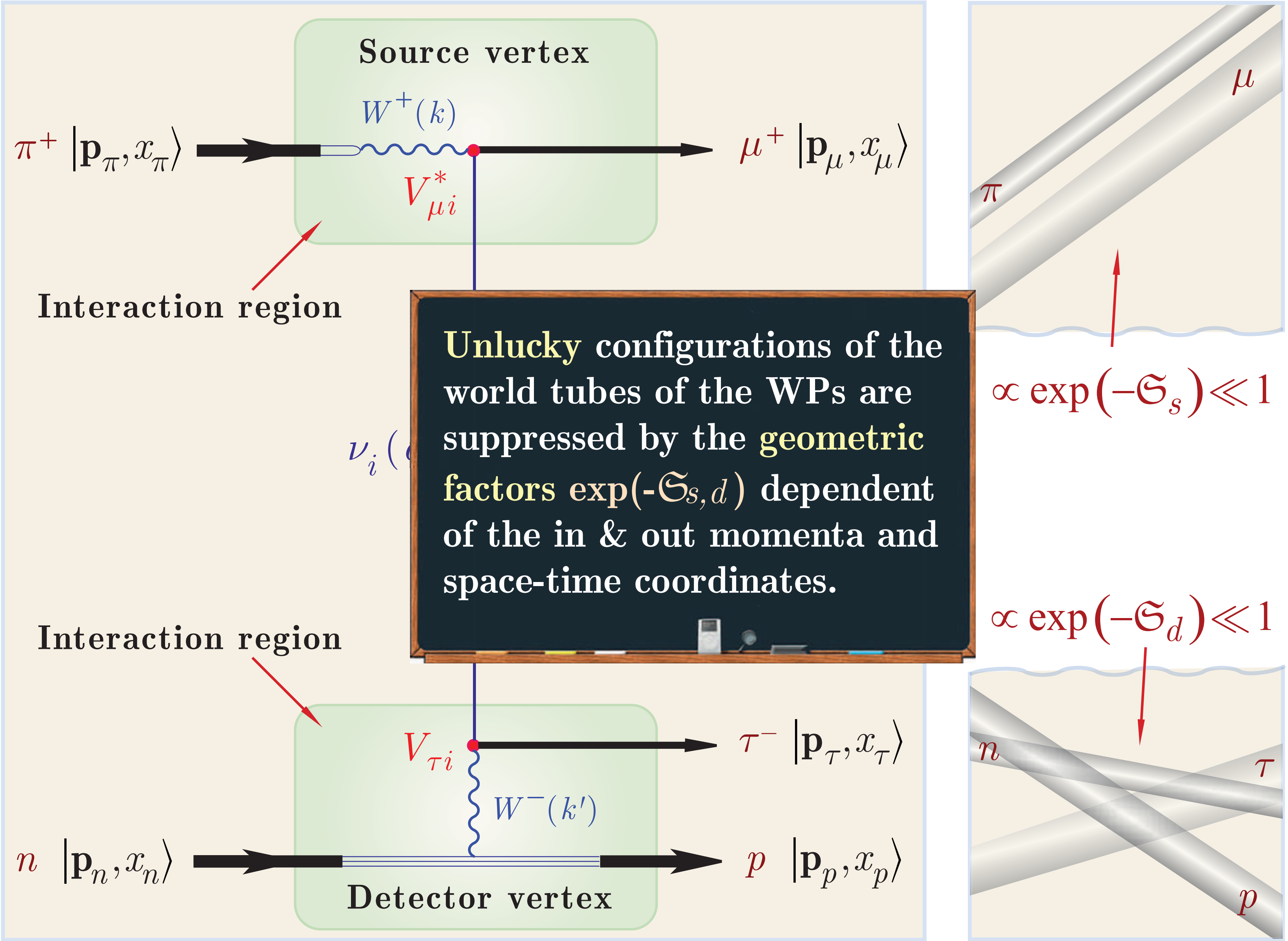

$\pi^+\ |\mathbf{p}_\pi\rangle$
$W^+(k)$
$\mu^+\ |\mathbf{p}_\mu\rangle$
$V^*_{\mu i}$
Feynman graphs with Fock legs cannot reproduce the $\nu$-oscillation phenomenon.
In the standard $S$ matrix perturbation theory the **in** & **out** states are **one-particle Fock states:**
$|\mathbf{p}_\varkappa\rangle = \sqrt{2E_{\mathbf{p}_\varkappa}}\, a^+_\varkappa(\mathbf{p}_\varkappa)\,|0\rangle$
$E_{\mathbf{p}_\varkappa} = \sqrt{\mathbf{p}^2_\varkappa + m^2_\varkappa}, \quad \varkappa = \pi, \mu, n, \ldots$
$\langle \mathbf{q}\,|\,\mathbf{k}\rangle = (2\pi)^3\, 2E_{\mathbf{k}}\,\delta\left(\mathbf{k}-\mathbf{q}\right)$
$V_{\tau i}$
$\tau^-\ |\mathbf{p}_\tau\rangle$
$W^-(k')$
$n\ |\mathbf{p}_n\rangle$
$p\ |\mathbf{p}_p\rangle$

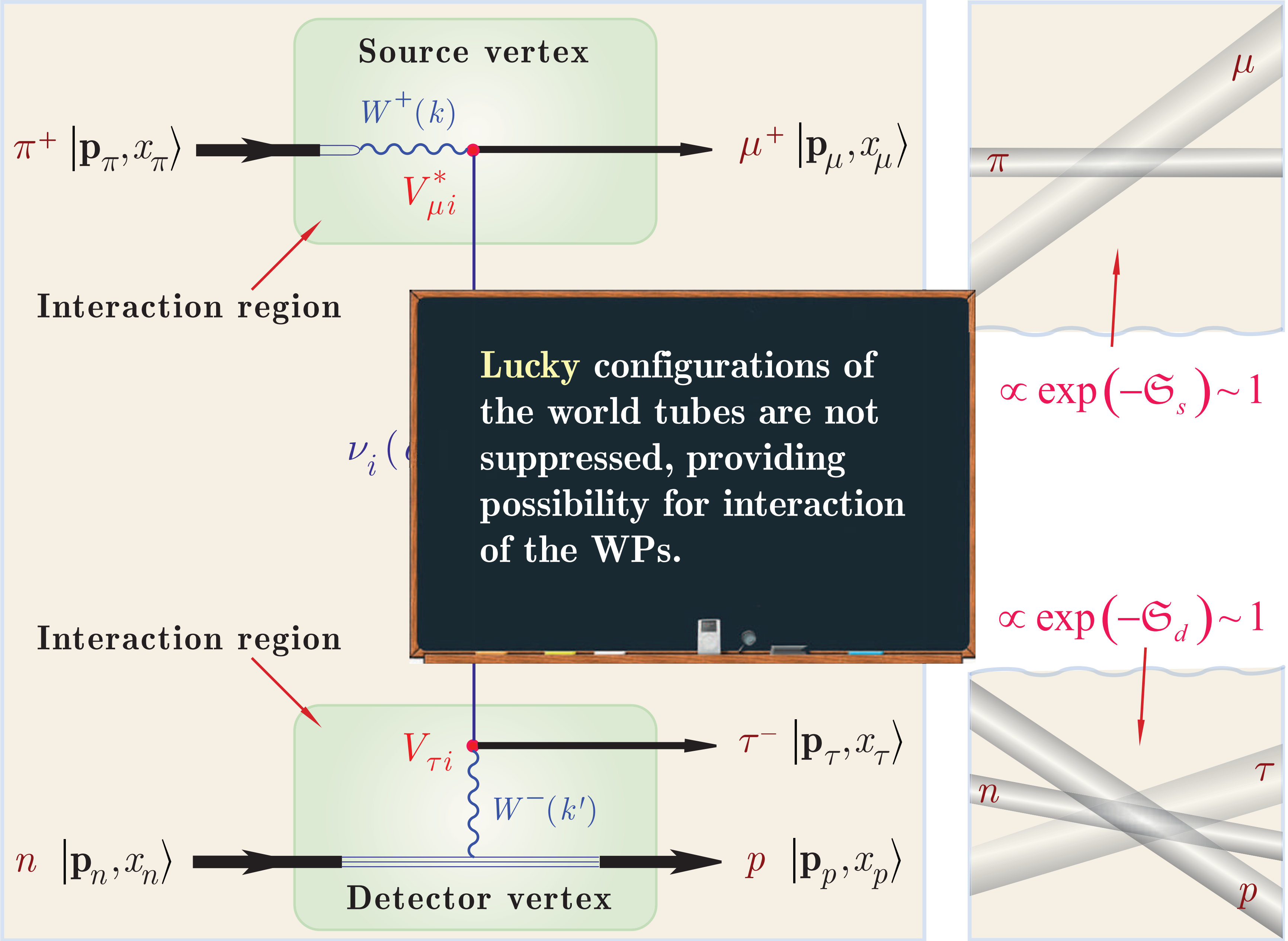


$\pi^+ \,|\mathbf{p}_\pi, x_\pi\rangle$
$W^+(k)$
$\mu^+ \,|\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
$\nu_i(q)$
$V_{\tau i}$
$\tau^- \,|\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n \,|\mathbf{p}_n, x_n\rangle$
$p \,|\mathbf{p}_p, x_p\rangle$
In our approach the in and out states are covariant wave packets:
$|\mathbf{p}_\varkappa, x_\varkappa\rangle = \sqrt{2E_{\mathbf{p}_\varkappa}}\, A^+_\varkappa(\mathbf{p}_\varkappa, x_\varkappa)\,|0\rangle$
$A^+_\varkappa(\mathbf{p}, x) = \int \frac{d\mathbf{k}\, \phi(\mathbf{k},\mathbf{p}) e^{i(k-p)x}}{2(2\pi)^3\sqrt{E_\mathbf{k} E_\mathbf{p}}} a^+_\varkappa(\mathbf{k})$
$A^+_\varkappa(\mathbf{p}, x) \overset{\mathrm{PWL}}{\longmapsto} a^+_\varkappa(\mathbf{p}) \Rightarrow \langle \mathbf{p}, x\,|\,\mathbf{p}, x\rangle = 2m\mathrm{V}_\star$

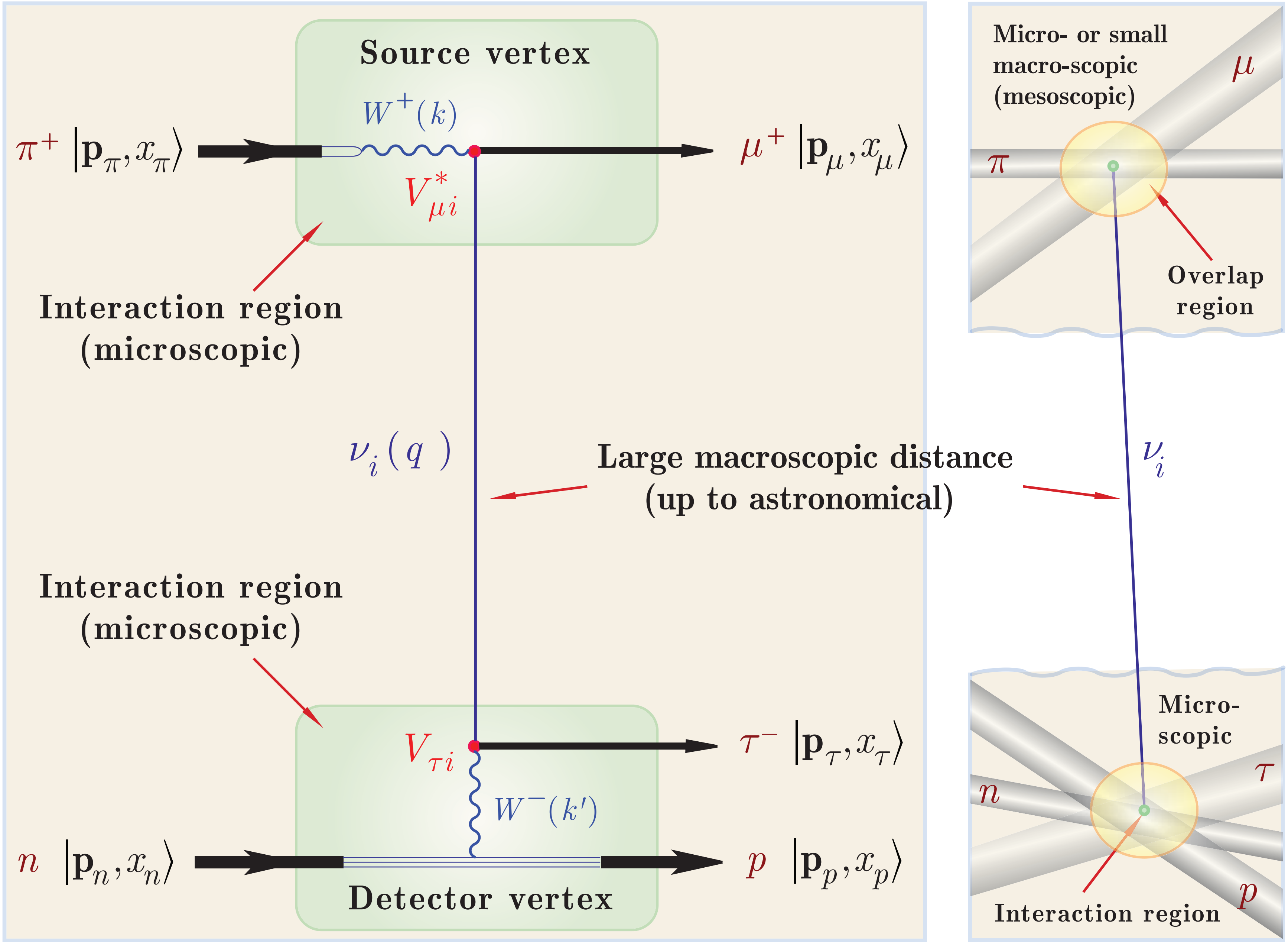

$\pi^+ \; |\mathbf{p}_\pi, x_\pi\rangle$
$W^+(k)$
$\mu^+ \; |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
For simplicity we omit the spin and other discrete variables in the WP states
In our approach the in and out states are covariant wave packets:
$|\mathbf{p}_\varkappa, x_\varkappa\rangle = \sqrt{2E_{\mathbf{p}_\varkappa}}\, A^+_\varkappa(\mathbf{p}_\varkappa, x_\varkappa)\,|0\rangle$
$A^+_\varkappa(\mathbf{p}, x) = \int \frac{d\mathbf{k}\, \phi(\mathbf{k},\mathbf{p}) e^{i(k-p)x}}{2(2\pi)^3\sqrt{E_\mathbf{k} E_\mathbf{p}}} a^+_\varkappa(\mathbf{k})$
$A^+_\varkappa(\mathbf{p}, x) \overset{\text{PWL}}{\longmapsto} a^+_\varkappa(\mathbf{p}) \Rightarrow \langle \mathbf{p}, x | \mathbf{p}, x\rangle = 2m\mathrm{V}_\star$
$V_{\tau i}$
$\tau^- \; |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n \; |\mathbf{p}_n, x_n\rangle$
$p \; |\mathbf{p}_p, x_p\rangle$

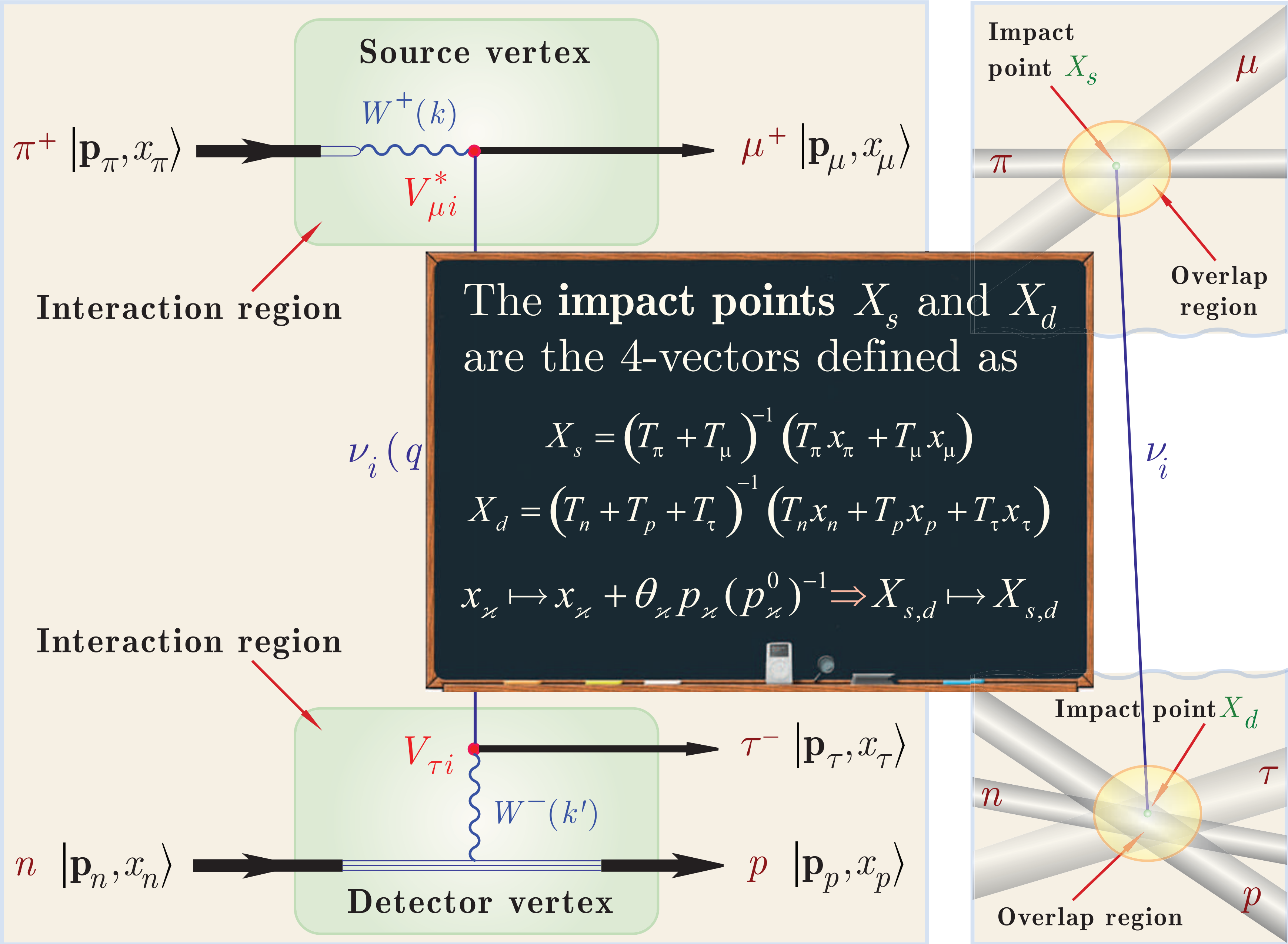

Source vertex
$W^+(k)$
$\pi^+ \; |\mathbf{p}_\pi, x_\pi\rangle$
$\mu^+ \; |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
Interaction region
$\nu_i(q)$
WP can be roughly thought as small interpenetrative cloudlets which are, however, much larger than the microscopic interaction regions in the source/detector vertices.
Interaction region
$V_{\tau i}$
$\tau^- \; |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n \; |\mathbf{p}_n, x_n\rangle$
$p \; |\mathbf{p}_p, x_p\rangle$
Detector vertex
$\pi$
$\mu$
$p$
$n$
$\tau$


Source vertex
$W^+(k)$
$\pi^+\ |\mathbf{p}_\pi, x_\pi\rangle$
$\mu^+\ |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
Interaction region
$\nu_i$
Unlucky configurations of the world tubes of the WPs are suppressed by the geometric factors exp(-$\mathfrak{S}_{s,d}$) dependent of the in & out momenta and space-time coordinates.
Interaction region
$V_{\tau i}$
$\tau^-\ |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n\ |\mathbf{p}_n, x_n\rangle$
$p\ |\mathbf{p}_p, x_p\rangle$
Detector vertex
$\pi$
$\mu$
$\propto \exp(-\mathfrak{S}_s) \ll 1$
$\propto \exp(-\mathfrak{S}_d) \ll 1$
$n$
$\tau$
$p$




**Lucky** configurations of the world tubes are not suppressed, providing possibility for interaction of the WPs.


Source vertex
$W^+(k)$
$\pi^+\ |\mathbf{p}_\pi, x_\pi\rangle$
$\mu^+\ |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
Interaction region
(microscopic)
$\nu_i(q)$
Large macroscopic distance
(up to astronomical)
Interaction region
(microscopic)
$V_{\tau i}$
$\tau^-\ |\mathbf{p}_\tau, x_\tau\rangle$
$W^-(k')$
$n\ |\mathbf{p}_n, x_n\rangle$
$p\ |\mathbf{p}_p, x_p\rangle$
Detector vertex
Micro- or small
macro-scopic
(mesoscopic)
$\mu$
$\pi$
Overlap
region
$\nu_i$
Micro-
scopic
$\tau$
$n$
$p$
Interaction region


Source vertex
$W^+(k)$
$\pi^+ \; |\mathbf{p}_\pi, x_\pi\rangle$
$\mu^+ \; |\mathbf{p}_\mu, x_\mu\rangle$
$V^*_{\mu i}$
Interaction region
$\nu_i\,(q$
Interaction region
$V_{\tau i}$
$W^-(k')$
$\tau^- \; |\mathbf{p}_\tau, x_\tau\rangle$
$n \; |\mathbf{p}_n, x_n\rangle$
$p \; |\mathbf{p}_p, x_p\rangle$
Detector vertex
Impact point $X_s$
$\mu$
$\pi$
Overlap region
$\nu_i$
Impact point $X_d$
$n$
$\tau$
$p$
Overlap region
The **impact points** $X_s$ and $X_d$ are the 4-vectors defined as
$X_s = (T_\pi + T_\mu)^{-1}(T_\pi x_\pi + T_\mu x_\mu)$
$X_d = (T_n + T_p + T_\tau)^{-1}(T_n x_n + T_p x_p + T_\tau x_\tau)$
$x_\varkappa \mapsto x_\varkappa + \theta_\varkappa p_\varkappa (p^0_\varkappa)^{-1} \Rightarrow X_{s,d} \mapsto X_{s,d}$

### 5.7.2 Space-time scales.

In the covariant WP approach there are several space-time scales:

- $T_I^{s,d}$ and $R_I^{s,d}$ – microscopic interaction time and radius defined by the Lagrangian.
- $T_O^{s,d}$ and $R_O^{s,d}$ – microscopic or small macroscopic dimensions of the overlap space-time regions of the interacting **in** and **out** packets in the source and detector vertices, defined by the effective dimensions of the packets.

  The suppression of the "unlucky" configurations of world tubes of the external packets is governed by the geometric factor in the amplitude:

  $$\exp\left[-\left(\mathfrak{S}_s + \mathfrak{S}_d\right)\right],$$

  where $\mathfrak{S}_{s,d}$ are the positive Lorentz and translation invariant functions of $\{\mathbf{p}_\varkappa\}$ and $\{x_\varkappa\}$. In the simplest one-parameter model of WP (relativistic Gaussian packet)

  $$\mathfrak{S}_{s,d} = \sum \sigma_\varkappa^2 \left|\boldsymbol{b}_\varkappa^\star\right|^2, \quad \varkappa \in S, D,$$

  where $\sigma_\varkappa$ are the momentum speeds of the packet $\varkappa$ and $\boldsymbol{b}_\varkappa^\star$ is the classical impact vector in the rest frame of the packet $\varkappa$ relative to the corresponding impact point.

- $T = X_d^0 - X_s^0$ and $L = |\mathbf{X}_d - \mathbf{X}_s|$ – large macroscopic neutrino time of flight and way between the impact points $X_s$ and $X_d$.
  For light neutrinos, the impact points lie very close to the light cone $T^2 = L^2$.
- In usual circumstance (terrestrial experiments) $T_I^{s,d} \ll T_O^{s,d} \ll T$ and $R_I^{s,d} \ll R_O^{s,d} \ll L$.

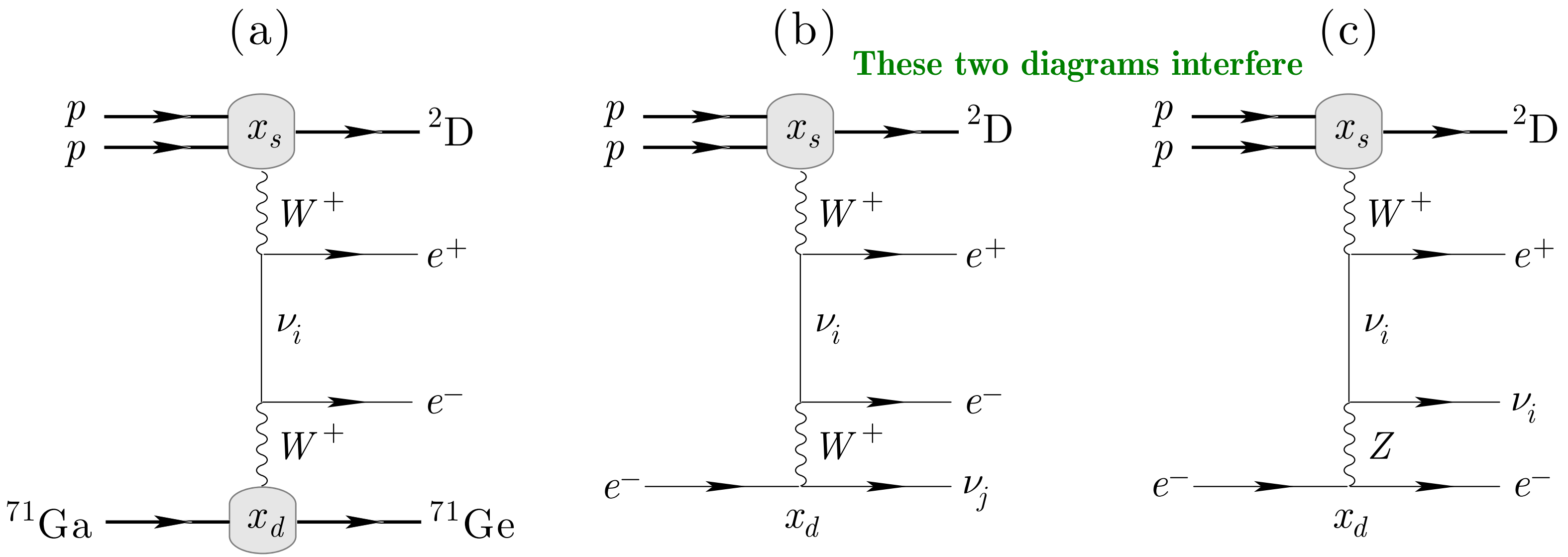
### 5.7.3 Examples of macroscopic diagrams.

- The $pp$ fusion.

The first reaction of the pp I branch

$$^{1}\text{H} + {}^{1}\text{H} \to {}^{2}\text{D} + e^{+} + \nu_e \quad (E_\nu < 420 \text{ keV})$$

lights the Sun and can be detected in Ga-Ge detectors like SAGE and GALLEX.



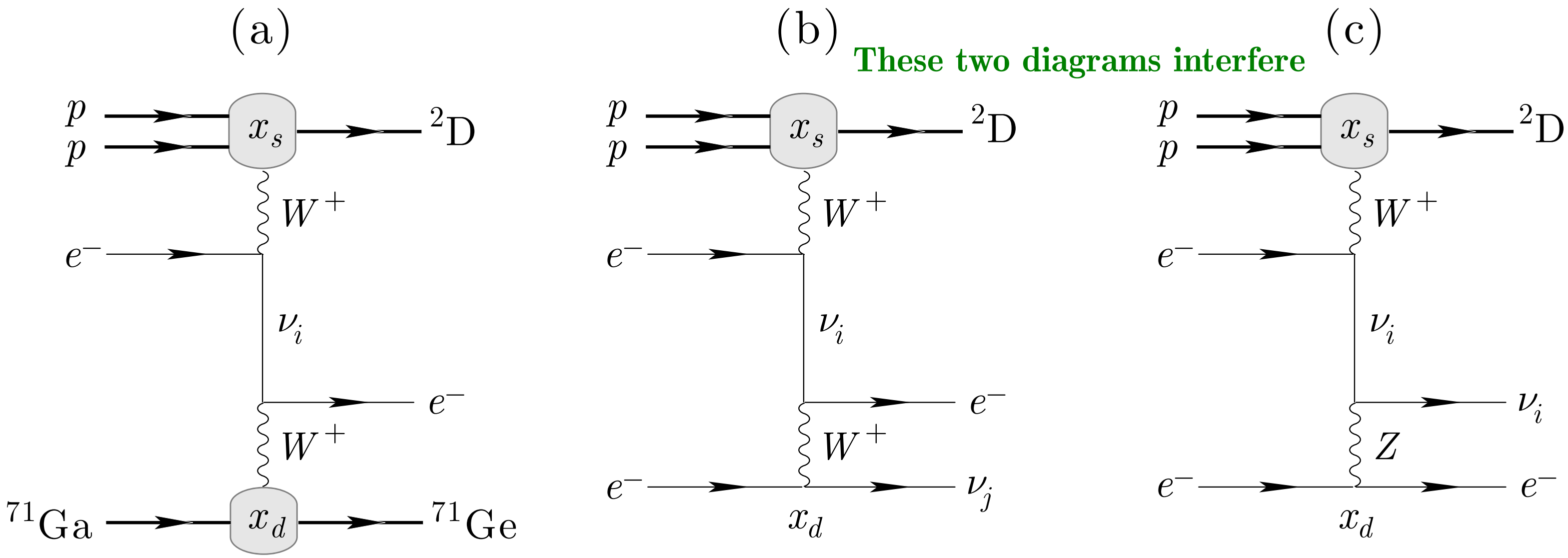

The Figure illustrates the detection of $pp$ neutrinos with gallium (a) and electron (b,c) targets. Unfortunately, the final electron energies in the reactions (b,c) are too low to be detected by Cherenkov method.

- The $pep$ fusion.

The reaction

$$^1\text{H} + {}^1\text{H} + e^- \to {}^2\text{D} + \nu_e \quad (E_\nu = 1.44 \text{ MeV})$$

accounts for about 0.25% of the deuterium created in the Sun in the $pp$ chain. It has a characteristic time scale $\sim 10^{12}$ yr that is larger than the age of the Universe. So it is *insignificant* in the Sun as far as energy generation is concerned. Enough $pep$ fusions happen to produce a detectable number of neutrinos in Ga-Ge detectors. Hence the reaction must be accounted for by those interested in the *solar neutrino problem*.




The Figure illustrates the detection of $pep$ neutrinos with gallium (a) and electron (b,c) targets. Similar to the $pp$ neutrino case, the diagram sets (c) and (d) interfere. While the final electron in the detector vertices of the diagrams (b,c) may have a momentum above the Cherenkov threshold, the current water-Cherenkov detectors SK and SNO+ are insensitive to the $pep$ neutrinos.

- The $\mu_{e3}$ decay

$$\mu^- \to e^- + \overline{\nu}_e + \nu_\mu$$

in the source can be detected through quasielastic scattering with production of $e^\pm$, $\mu^\pm$, or $\tau^\pm$; of course, only $\mu^\pm$ production is permitted in SM. The diagrams (a) and (b) are for both Dirac and Majorana (anti)neutrinos, while diagrams (c) and (d) are only for Majorana neutrinos.

---

In the Majorana case, the diagrams (a), (d) and (b), (c) interfere. Potentially this provides a way for distinguishing between the Dirac and Majorana cases. Unfortunately, the diagrams (c) and (d) are suppressed by a factor $\propto m_i/E_\nu$.

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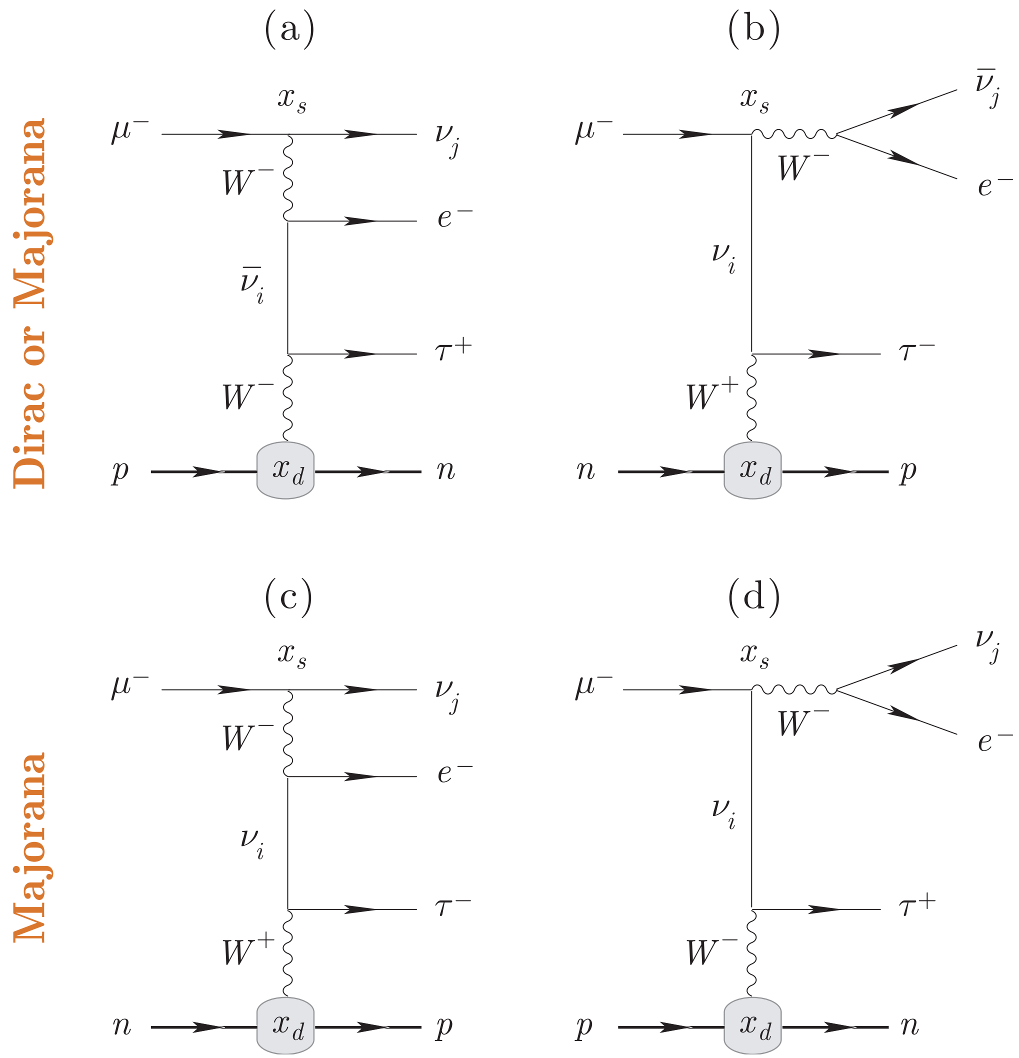
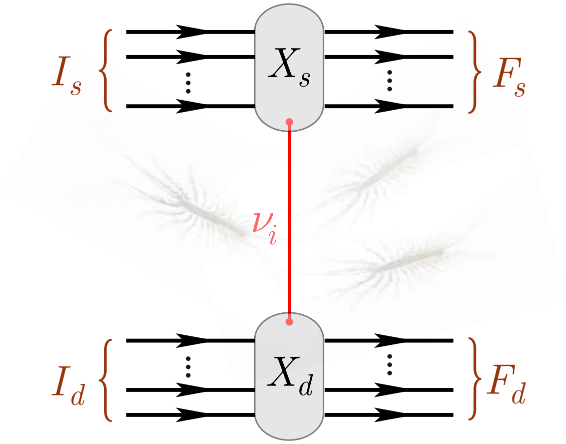
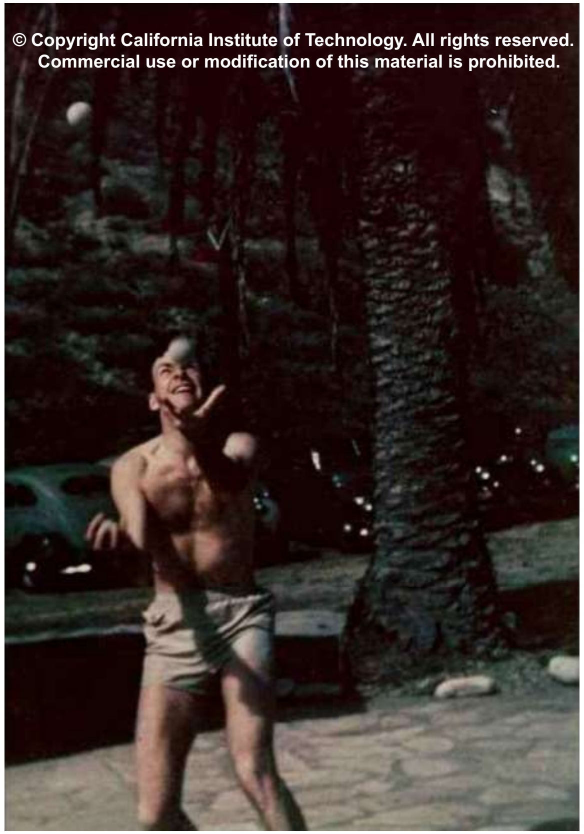



Similar diagrams can be drawn for $\tau_{e3}$ and $\tau_{\mu3}$ decays.

## 5.8 Shortest summary.

The QFT-based neutrino oscillation theory deals with generic Feynman's macrodiagrams ("myriapods"). ▷
The external legs correspond to asymptotically free incoming ("in") and outgoing ("out") wave packets (WP) in the coordinate representation. Here and below: $I_s$ ($F_s$) is the set of in (out) WPs in $X_s$ ("source"), $I_d$ ($F_d$) is the set of in (out) WPs in $X_d$ ("detector").




The internal line denotes the causal Green's function of the neutrino mass eigenfield $\nu_i$ ($i = 1, 2, 3, \ldots$). The blocks (vertices) $X_s$ and $X_d$ must be macroscopically separated in space-time. This explains the term "macroscopic Feynman diagram".
For narrow WPs, the Feynman rules in the formalism are to be modified[a] in a rather trivial way: for each external line, the *standard* (plain-wave) factor must be multiplied by

$$\begin{cases} e^{-ip_a(x_a-x)}\psi_a\left(\mathbf{p}_a, x_a - x\right) & \text{for } a \in I_s \oplus I_d, \\ e^{+ip_b(x_b-x)}\psi_b^*\left(\mathbf{p}_b, x_b - x\right) & \text{for } b \in F_s \oplus F_d, \end{cases} \tag{16}$$

where each function $\psi_\varkappa\left(\mathbf{p}_\varkappa, x\right)$ ($\varkappa = a, b$) is specified by the mass $m_\varkappa$ and momentum spread $\sigma_\varkappa$. The lines inside $X_s$ and $X_d$ (including possible loops) and vertex factors remain unchanged.

[a] For non-commercial purposes.

### 5.8.1 Important class of macrodiagrams.

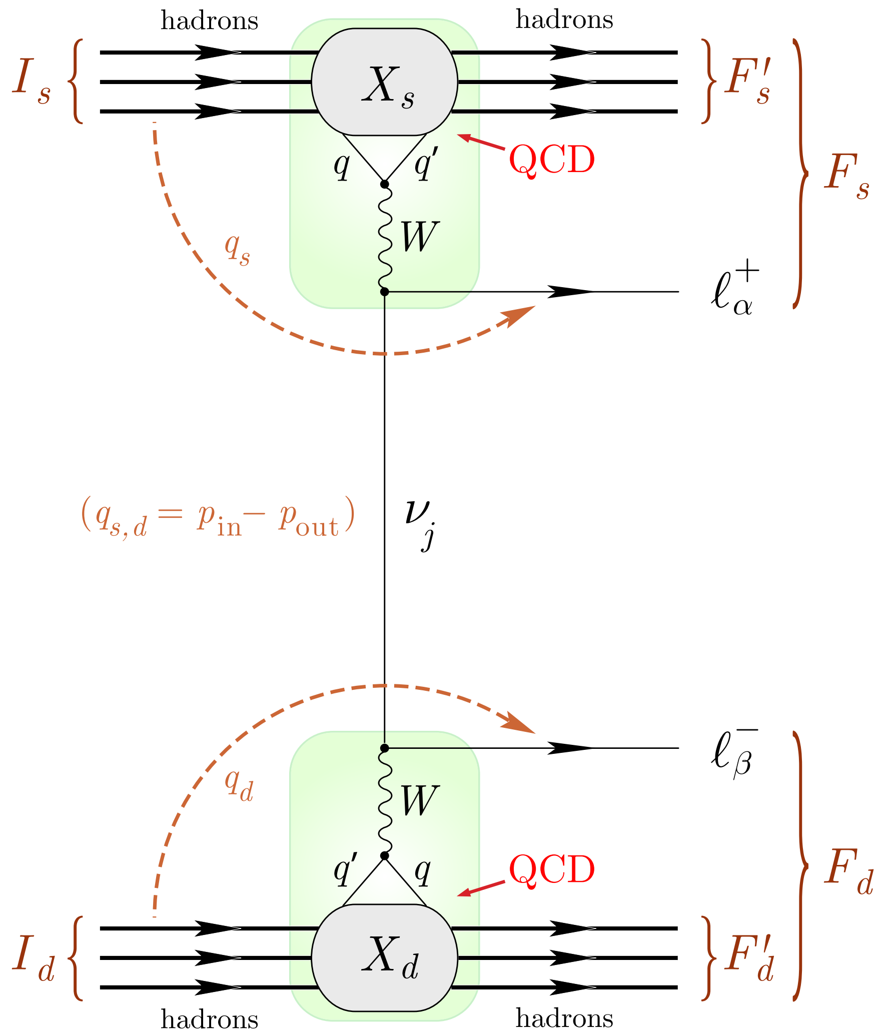
As a practically important example, we consider the charged-current induced production of charged leptons $\ell_\alpha^+$ and $\ell_\beta^-$ ($\ell_{\alpha,\beta} = e, \mu, \tau$) in the process

$$I_s \oplus I_d \to F_s' + \ell_\alpha^+ \oplus F_d' + \ell_\beta^-, \qquad (17)$$

We assume for definiteness that all the external substates $I_s$, $I_d$, $F_s'$, and $F_d'$ consist exclusively of (asymptotically free) hadronic WPs.

Consequently, if $\alpha \neq \beta$, the process (17) violates the lepton numbers $L_\alpha$ and $L_\beta$ that is only possible via exchange of massive neutrinos (no matter whether they are Dirac or Majorana particles).

In the lowest nonvanishing order in electroweak interactions, the process (17) is described by the sum of the diagrams shown in the figure. ▷




The impact points $X_s$ and $X_d$ in the figure are macroscopically separated and the asymptotic conditions are assumed to be fulfilled.

### 5.8.2 Main result.

A rather general (while not the most general) expression for the number of neutrino-induced events corresponding to the diagram shown in previous page, is of the form

$$\frac{N_{\beta\alpha}}{\tau_d} = \sum_{\text{spins}} \int d\mathbf{x} \int d\mathbf{y} \int d\mathfrak{P}_s \int d\mathfrak{P}_d \int d|\mathbf{q}| \frac{\mathcal{P}_{\alpha\beta}\left(|\mathbf{q}|, |\mathbf{y}-\mathbf{x}|\right)}{4(2\pi)^3|\mathbf{y}-\mathbf{x}|^2},$$

$$\mathcal{P}_{\alpha\beta}\left(|\mathbf{q}|, |\mathbf{y}-\mathbf{x}|\right) = \sum_{ij} V_{\beta j} V_{\alpha i} V_{\beta i}^* V_{\alpha j}^* \exp\left(i\varphi_{ij} - \mathcal{A}_{ij}^2 - \mathcal{C}_{ij}^2 - \varTheta_{ij}\right) S_{ij},$$

$$S_{ij} = \frac{\exp(-\mathcal{B}_{ij}^2)}{4\mathfrak{D}\tau_d} \sum_{l,l'=1}^{2} (-1)^{l+l'+1} \mathsf{lerf}\left[2\mathfrak{D}\left(x_l^0 - y_{l'}^0 + |\mathbf{y}-\mathbf{x}|\right) + i\mathcal{B}_{ij}\right],$$

$$\mathfrak{D} = 1/\sqrt{2\widetilde{\Re}^{\mu\nu} l_\mu l_\nu},$$

$$d\mathfrak{P}_s = (2\pi)^4 \delta_s(q - q_s)|M_s|^2 \prod_{a\in I_s} \frac{d\mathbf{p}_a f_a(\mathbf{p}_a, s_a, x)}{(2\pi)^3 2E_a} \prod_{b\in F_s} \frac{d\mathbf{p}_b}{(2\pi)^3 2E_b},$$

$$d\mathfrak{P}_d = (2\pi)^4 \delta_d(q + q_d)|M_d|^2 \prod_{a\in I_d} \frac{d\mathbf{p}_a f_a(\mathbf{p}_a, s_a, y)}{(2\pi)^3 2E_a} \prod_{b\in F_d} \frac{[d\mathbf{p}_b]}{(2\pi)^3 2E_b}.$$

The ingredients are listed on p. 96. These formulas do not take into account the inverse-square law violation corrections, for which we unfortunately do not have enough time to discuss.[a]

[a] See VN & D. S. Shkirmanov, Eur. Phys. J. C **73** (2013) 2627; Universe **7** (2021) 246 and refs. therein.

Таблица 1: Ingredients of the equations shown in p. 95, in the leading order for the off-mass-shell (short distances) and on-mass-shell (long distances) regimes. Here $L = |\mathbf{y} - \mathbf{x}|$, $\Delta m_{ij}^2 = m_i^2 - m_j^2$, $\mathcal{Q}^4 = \left(\mathcal{R}^{00}\mathcal{R}^{\mu\nu} - \mathcal{R}^{0\mu}\mathcal{R}^{0\nu}\right) l_\mu l_\nu$, $Y^\mu = \widetilde{\Re}_s^{\mu\nu} q_{s\nu} - \widetilde{\Re}_d^{\mu\nu} q_{d\nu}$, $\widetilde{\Re}_{s,d}$ are the so-called inverse overlap tensors of in and out WPs in the source and detector vertices, $\widetilde{\Re} = \widetilde{\Re}_s + \widetilde{\Re}_d$, $\mathcal{R}$ is the tensor inverse to $\widetilde{\Re}$ (that is $\mathcal{R}^{\mu\lambda}\widetilde{\Re}_{\lambda\nu} = \delta_\nu^\mu$), and $\Sigma = \det(\mathcal{R})^{1/8}$ is the scale of the energy-momentum dispersion of the effective neutrino WP. Last column shows the order of magnitude (OoM) of the quantity. Evidently, $E_\nu \simeq q_0 \simeq |\mathbf{q}|$ in the UR approximation.

| **Quantity** | **Off-shell regime** | **On-shell regime** | **OoM** |
|---|---|---|---|
| $\varphi_{ij}$ | $\dfrac{\Delta m_{ij}^2 L}{2\|\mathbf{q}\|}$ | $\dfrac{\Delta m_{ij}^2 L}{2E_\nu}$ | $\dfrac{\|\Delta m_{ij}^2\| L}{E_\nu}$ |
| $\mathcal{A}_{ij}^2$ | $\left(\dfrac{\Delta m_{ij}^2 L}{2\|\mathbf{q}\|^2}\right)^2 \dfrac{\mathcal{Q}^4}{2\mathcal{R}^{\mu\nu} l_\mu l_\nu}$ | $\left(\dfrac{\Delta m_{ij}^2 L}{2E_\nu^2}\right)^2 \dfrac{1}{2\widetilde{\Re}^{\mu\nu} l_\mu l_\nu}$ | $\left(\dfrac{\Delta m_{ij}^2}{E_\nu^2}\Sigma L\right)^2$ |
| $\mathcal{B}_{ij}$ | $\dfrac{\Delta m_{ij}^2}{4\|\mathbf{q}\|}\sqrt{\dfrac{\widetilde{\Re}^{\mu\nu} l_\mu l_\nu}{2}}\,\dfrac{\mathcal{R}^{0\mu} l_\mu}{\mathcal{R}^{\mu\nu} l_\mu l_\nu}$ | $\dfrac{\Delta m_{ij}^2}{4E_\nu}\sqrt{\dfrac{\widetilde{\Re}^{\mu\nu} l_\mu l_\nu}{2}}\,\dfrac{Y_k l_k}{Y^\mu l_\mu}$ | $\dfrac{\|\Delta m_{ij}^2\|}{\Sigma E_\nu}$ |
| $\mathcal{C}_{ij}^2$ | $\left(\dfrac{\Delta m_{ij}^2}{2\|\mathbf{q}\|}\right)^2 \dfrac{1}{8\mathcal{R}^{\mu\nu} l_\mu l_\nu}$ | $0$ | $\left(\dfrac{\Delta m_{ij}^2}{\Sigma E_\nu}\right)^2$ |
| $\Theta_{ij}$ | $\dfrac{m_i^2 + m_j^2}{4\|\mathbf{q}\|}\left[\widetilde{\Re}_s^{0\mu}\left(q - q_s\right)_\mu + \widetilde{\Re}_d^{0\mu}\left(q + q_d\right)_\mu\right]$ | $\dfrac{m_i^2 + m_j^2}{4q_0}\left[\widetilde{\Re}_s^{\mu k} l^k \left(q_0 l - q_s\right)_\mu + \widetilde{\Re}_d^{\mu k} l^k \left(q_0 l + q_d\right)_\mu\right]$ | $\dfrac{m_i^2 + m_j^2}{\Sigma E_\nu}$ |