Renormalization group analysis of a self-organized critical system: Intrinsic anisotropy opposed to random medium

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Systems with self-organised criticality & turbulence (1)

What is interesting in models of self-organized criticality (SOC)?

- they are open nonequilibrium systems with dissipative transport;
- they are believed to be ubiquitous in nature [1];
- they arrive at their critical states due to their intrinsic dynamics, i.e. they have no tuning parameter.

Self-organised critical systems under the influence of turbulence can be studied by renormalization group method!

Figure: The SOC models are often found in nature

Purpose of the study (2)

The goal of our research is to study universality classes (types of critical behavior) of a system with self-organised criticality described by anisotropic continuos model of a "running sandpile" [2] while taking into account turbulent motion of the environment.

The method (3)

Stochastic problem → Field theoretic formulation (the De Dominicis-Janssen action functional [3] → Analysis of canonical dimensions → Feynman diagrams calculation → Renormalization equations → Critical exponents

Description of the model (4)

The model of a self-organised critical system behavior is continuous equation for height transport with strong anisotropy (the Hwa-Kardar equation – HK) [2]:

$$\partial_t h = \nu_\perp \partial_\perp^2 h + \nu_\parallel \partial_\parallel^2 h - \partial_\parallel h^2/2 + f. \tag{1}$$

- h is a height of the profile; $\nu_\perp, \nu_\parallel > 0$ are viscosity coefficients;
- $\mathbf{x} = \mathbf{x}_\perp + \mathbf{n}x_\parallel$, $|\mathbf{n}| = 1$, $\mathbf{x}_\perp \mathbf{n} = 0$, d is the dimension of the $\mathbf{x}$ space, $\partial_\perp$, $\partial_\parallel$ are transverse and longitudinal derivatives respectively;
- $f = f(x)$ is the Gaussian random noise with zero mean:

$$\langle f(x) f(x') \rangle = C_0 \delta(t-t') \delta^{(d)}(\mathbf{x}-\mathbf{x}');\ C_0 = g \nu_\perp^{3/2} \nu_\parallel^{3/2}$$

The turbulent motion of the environment is modeled by the Navier-Stokes equation with an external random force (isotropic incompressible viscous fluid):

$$\partial_t v_i + (\boldsymbol{v} \cdot \boldsymbol{\partial}) v_i = \nu_0 \partial^2 v_i - \partial_i P + \eta_i. \tag{2}$$

P is the pressure, η_i is the transverse external random force per unit mass, ν_0 is the kinematic coefficient of viscosity. η_i is a Gaussian statistics with zero mean, prescribed pair covariance with vanishing correlation time:

$$\langle \eta_i(t,\mathbf{x}) \eta_j(t',\mathbf{x}') \rangle = \frac{\delta(t-t')}{(2\pi)^d} \int P_{ij}(\mathbf{k})\, d_v(k) \exp i\mathbf{k}(\mathbf{x}-\mathbf{x}')\, d\mathbf{k},$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector,

$$d_v(k) = D_1 + D_2 k^{4-d-\xi}$$

- Also the velocity field $v_i(x)$ is introduced by the replacement $\partial_t h \to \nabla_t h \equiv \partial_t h + (v_i \partial_i) h$.
- $\partial_i v_i = 0$; $D_1, D_2 > 0$ are amplitude factors.

Field theoretic formulation of the model (5)

The stochastic problem (1) is equivalent to the field theoretic model with the action functional

$$\mathcal{S}\{h, h', \boldsymbol{v}\} = C_0 h'h'/2 + D_0 v'v'/2 + h'\{-\partial_t h - (v\partial)h + \nu_{\parallel 0} \partial_\parallel^2 h + \nu_{\perp 0} \partial_\perp^2 h - \partial_\parallel h^2/2\} + v'\{-\partial_t v - (v\partial)v + \nu_0 \partial^2 v\},$$

where $D_0 \equiv d_v(k)$.
The model has three interaction vertices: $-v'(v\partial)v$, $-h'\partial_\parallel h^2/2$, $-h'(v\partial)h$ (**Note**: h' is always under ∂)

Diagrammatic representation (6)

- We will denote the model propagators $\langle hh \rangle_0$ as a straight line, $\langle hh' \rangle_0$ as a straight line with a small stroke and similarly for the velocity propagators $\langle vv \rangle_0$ and $\langle vv' \rangle_0$, but instead of straight lines - wavy ones.
- The coupling constants are $g_0 = C_0/(\nu_{\parallel 0}\nu_{\perp 0})^{3/2}$, $w_0 = D_{10}/\nu_0$, $x_{10} = \nu_{\parallel 0}/\nu_0$ and $x_{20} = \nu_{\perp 0}/\nu_0$ (if $D_{20} = 0$).

Avaliable symmetries (7)

- The Galilean symmetry of the problem augmented with the velocity field: $\mathbf{v} \to \mathbf{v} - \mathbf{n}u$, $u = const$.
- Although the symmetry of the original HK equation (1) is not performed ($h \to h - u$, $u = const$) in the full model, it reduce the number of such counter terms as, for example, $< hhh' >$.

The consequence:

- canonical dimensions analysis coupled with the symmetries proves that our model is **multiplicatively renormalizable**.

Conclusion (10)

- Was constructed and renormalized a field theory equivalent to the original problem.
- The point of the pure turbulent advection is IR attractive for the most realistic values of the spatial dimension d = 2 and d = 3. This means that isotropic motion "dominates" over the nonlinearity and the anisotropy at those values.
- In the future, work from will continue for $D_2 \neq 0$.

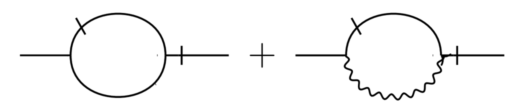
Renormalization (8)

- Renormalized action functional:

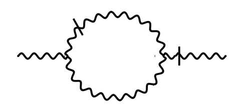
$$\mathcal{S}_R = Ch'h'/2 + Dv'v'/2 + h'\{-\partial_t h - (v\partial)h + Z_1 \nu_\parallel \partial_\parallel^2 h + Z_2 \nu_\perp \partial_\perp^2 h - \partial_\parallel h^2/2\} + v'\{-\partial_t v - (v\partial)v + Z_3 \nu \partial^2 v\}$$

- For Z_1, Z_2 and Z_3 the calculation is done to the first order of the expansion in $\varepsilon = 4 - d$ (one-loop approximation) only for $D_2 = 0$:

$$\langle hh' \rangle_{1-\text{ir}} = \mathrm{i}\omega - \nu_\parallel p_\parallel^2 Z_1 - \nu_\perp p_\perp^2 Z_2 +$$



$$\langle vv' \rangle_{1-\text{ir}} = \mathrm{i}\omega - \nu p^2 Z_3 +$$



$$Z_1 = 1 - \frac{1}{\varepsilon}\left[g\frac{3}{16} + \frac{w}{2\,x_1\left(\sqrt{1+x_1}+\sqrt{1+x_2}\right)^2}\left(1 + 2\sqrt{\frac{1+x_1}{1+x_2}}\right)\right] \equiv 1 - \frac{1}{\varepsilon}\left[g\frac{3}{16} + w f_1(x_1,x_2)\right],$$

$$Z_2 = 1 - \frac{w}{6\,\varepsilon\, x_2\left(\sqrt{1+x_1}+\sqrt{1+x_2}\right)^2}\left(5 + 4\sqrt{\frac{1+x_1}{1+x_2}}\right) \equiv 1 - \frac{w}{\varepsilon} f_2(x_1,x_2)\ ,\ Z_3 = 1 - \frac{1}{8\pi^2\varepsilon}\frac{w}{8}.$$

- Other terms in the renormalized action functional are finite due to the Galilean symmetry, closed circuits of retarded propagators, presence of a transverse projector.
- Were calculated RG functions, beta functions, which took the following form:

$$\beta_g = -g\left[\varepsilon - \frac{3}{2}\gamma_1 - \frac{3}{2}\gamma_2\right],\ \beta_w = -w\left[\varepsilon - 3\gamma_3\right],\ \beta_{x_1} = -x_1\left[\gamma_1 - \gamma_3\right],\ \beta_{x_2} = -x_2\left[\gamma_2 - \gamma_3\right],$$

where $\gamma_i = \widetilde{\mathcal{D}}_\mu \ln Z_i$, $\widetilde{\mathcal{D}}_\mu = \mathcal{D}_\mu + \beta_g\partial_g + \beta_w\partial_w + \beta_{x_1}\partial_{x_1} + \beta_{x_2}\partial_{x_2} - \gamma_\nu \mathcal{D}_\nu$ is the differential operator,

$$\gamma_1 = g\frac{3}{16} + w f_1(x_1,x_2), \quad \gamma_2 = w f_2(x_1,x_2), \quad \gamma_3 = \frac{w}{8}.$$

Fixed points and scaling regimes (9)

The RG equation for this model:

$$\left(D_\mu + \beta_g\partial_g + \beta_w\partial_w + \beta_{x_1}\partial_{x_1} + \beta_{x_2}\partial_{x_2} - \gamma_{\widetilde{\nu}} D_{\widetilde{\nu}} + n_{\boldsymbol{v}}\gamma_{\boldsymbol{v}} + n_{\boldsymbol{v}'}\gamma_{\boldsymbol{v}'} + n_h\gamma_h + n_{h'}\gamma_{h'}\right) W^R = 0, \tag{3}$$

where n_i – number of corresponding fields, W^R– a Green's function.
The canonical (momentum and frequency) scale equations:

$$\begin{gathered}\Big(D_\mu - D_x + d^p_{\widetilde{\nu}} D_{\widetilde{\nu}} + d^p_g D_g + d^p_w D_w + d^p_{x_1} D_{x_1} + d^p_{x_2} D_{x_2} - n_{\boldsymbol{v}} d^p_{\boldsymbol{v}} - n_{\boldsymbol{v}'} d^p_{\boldsymbol{v}'} - n_h d^p_h - n_{h'} d^p_{h'}\Big) W^R = 0,\\ \Big(-D_t + d^\omega_{\widetilde{\nu}} D_{\widetilde{\nu}} - n_{\boldsymbol{v}} d^\omega_{\boldsymbol{v}} - n_{\boldsymbol{v}'} d^\omega_{\boldsymbol{v}'} - n_h d^\omega_h - n_{h'} d^\omega_{h'}\Big) W^R = 0,\end{gathered} \tag{4}$$

where d_i– canonical dimension of corresponding parameter or field.
The equation for critical IR scaling:

$$\Big(-D_x - \Delta_\omega D_t + d^p_g D_g + d^p_w D_w + d^p_{x_1} D_{x_1} + d^p_{x_2} D_{x_2} - n_{\boldsymbol{v}}\Delta_{\boldsymbol{v}} - n_{\boldsymbol{v}'}\Delta_{\boldsymbol{v}'} - n_h\Delta_h - n_{h'}\Delta_{h'}\Big) W^R = 0. \tag{5}$$

- The Gaussian fixed point:
  $g_* = 0, w_* = 0$; $x_1 \neq 0$ and $x_2 \neq 0$ are any positive numbers,
  $\lambda_i = \{0, 0, -\varepsilon, -\varepsilon\}$– IR attractive for $\varepsilon < 0$. Critical exponents: $\Delta_h = \Delta_v = 1$, $\Delta_{h'} = \Delta_{v'} = d - 1$;
- Curved line of fixed points:

$$w_* = \varepsilon\frac{8}{3},\ f_2(x_1^*, x_2^*) = \frac{1}{8},\ g_* = \frac{128}{9}\varepsilon\left(\frac{1}{8} - f_1(x_1^*, x_2^*)\right), \quad x_2^* \in \left(0, \frac{\sqrt{13}-1}{2}\right]$$

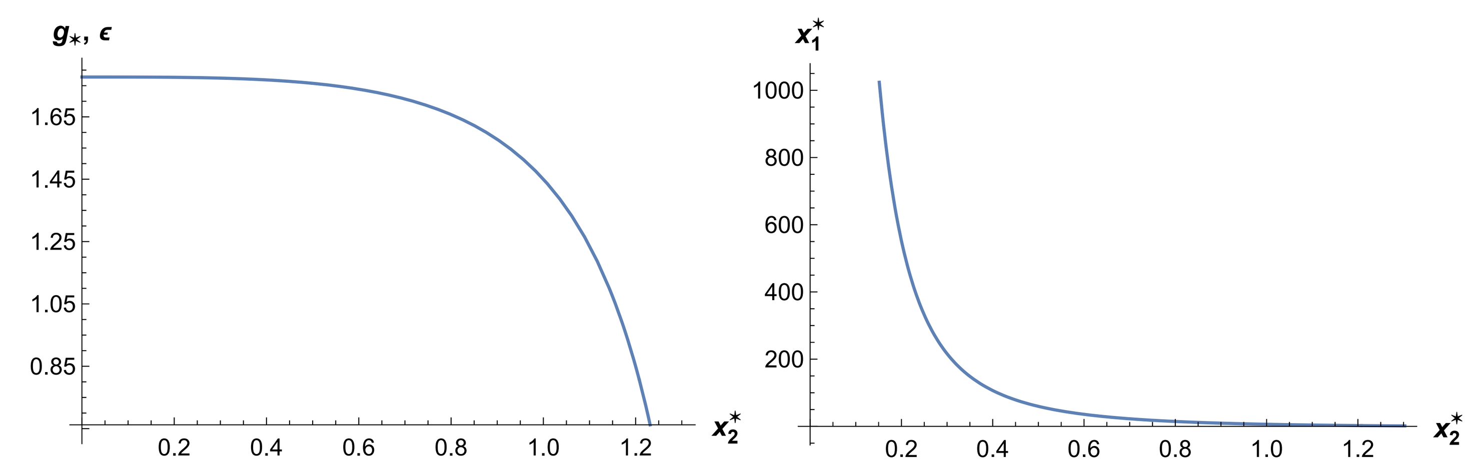


Figure: Charges g_* and x_1^* parameterized by x_2^*

$\lambda_i = \{0, \varepsilon, \lambda_3, \lambda_4\}$– IR attractive for $\varepsilon > 0$.
The equation for $\lambda_{3,4}$:

$$\lambda^2 + \lambda\left[\frac{8\varepsilon}{3}\left(x_1\frac{\partial f_1}{\partial x_1} + x_2\frac{\partial f_2}{\partial x_2}\right) - \frac{9}{32}g\right] + \frac{64\varepsilon^2}{9}x_1x_2\left(\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\right) + \frac{3\varepsilon}{4}g\left(x_1\frac{\partial f_2}{\partial x_1} - x_2\frac{\partial f_2}{\partial x_2}\right) = 0.$$

where $g = g_*, x_1 = x_1^*, x_2 = x_2^*$.

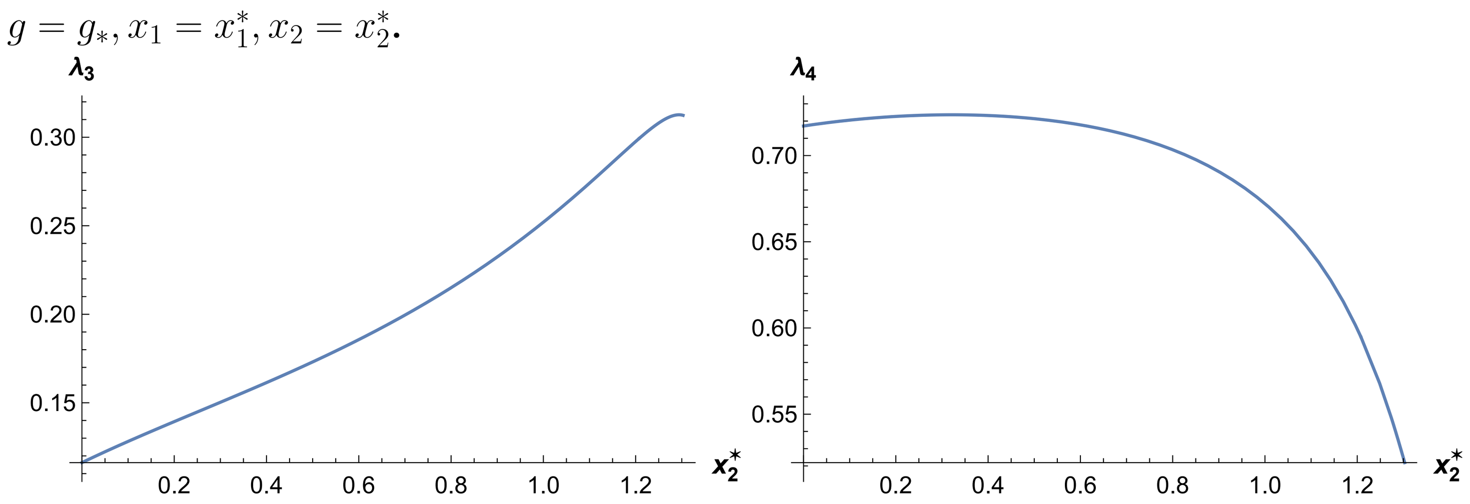


Figure: Eigenvalues λ_3 and λ_4 normalized to ε and parameterized by x_2^*

Critical exponents: $\Delta_h = \Delta_v = 1 - \varepsilon/3$, $\Delta_{h'} = \Delta_{v'} = d - 1 + \varepsilon/3$.

- The point of the pure turbulence advection:

$$w_* = 8\epsilon/3,\ g_* = 0,\ x_1^* = x_2^* = \frac{\sqrt{13}-1}{2}.$$

Unstable points were found in other systems: $y_{1,2} = x_{1,2}^{-1};\ u_{1,2} = w\, x_{1,2}^{-1};\ u = w\, x_1^{-1} x_2^{-1}$.
For example: $g_* = 32\varepsilon/9, w_* = 0, y_1^* = 0, \forall y_2^*,\quad \lambda_i = \{0, -\varepsilon, 2\varepsilon/3, \varepsilon\}$. This point belongs to the class of universality of the pure Hwa-Kardar equation without turbulent motion of the medium.

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