Lyapunov Growth in Nonlinear Vector Mechanics

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Abstract

We compute classical and quantum Lyapunov exponents for vector mechanics with broken $O(N)$ symmetry. In high temperature limit Lyapunov exponents approximately coincide and scale as $\kappa \approx 1.3\sqrt[4]{\lambda T}/N$ with temperature T, number of degrees of freedom N, and coupling constant λ.

Classical Lyapunov exponents

Consider Hamiltonian system, $z^I = (x^i, p_j)$

$$\dot{z}^I = \pi^{IJ}\frac{\partial H}{\partial z^J} \quad \to \quad z = z(t; z_0)$$

How to characterize deviation of neighboring trajectories?

- Intuitively

$$\|z(t; z_0) - z(t; z_0 + \Delta z)\| \sim e^{\kappa t} \leq L_{\text{system}}$$

- Rigorously. Define matrix Φ

$$\Phi^{IJ}(t; z_0) := \frac{\partial z^I(t; z_0)}{\partial z_0^J} \quad \to \quad \dot{\Phi}^{IJ} = \pi^{IK}\frac{\partial^2 H}{\partial z^K \partial z^L}\Phi^{LJ}$$

  SVD decompose it $\Phi = U\,\Sigma\,V^T$, $\Sigma = \mathrm{diag}(\sigma_{\text{max}}, \dots \sigma_{\text{min}})$ and define Lyapunov exponent κ as

$$\sigma_{\text{max}} \sim e^{\kappa t} \quad \to \quad \boxed{\kappa := \lim_{t\to\infty}\frac{1}{t}\log\sigma_{\text{max}}(t)}$$

But! Lyapunov exponent depends on initial conditions

$$\kappa = \kappa(z_0)$$

To make it independent, just average it

$$\kappa(z_0) \mapsto \kappa = \langle\kappa(z_0)\rangle_{z_0} = \int d\mu(z_0)\,\kappa(z_0),$$

e.g. with measure

$$d\mu_\beta(z) \sim dz\, e^{-\beta H(z)}, \quad d\mu_E(z) \sim dz\,\delta(H(z) - E)$$

For large N models first measure can be estimated by second one in $\mathcal{O}(1/\sqrt{N})$ order.

OTOCs and quantum Lyapunov exponents

How to quantize the notion of Lyapunov exponent? Write defitition of Φ as

$$\Phi^{IJ} = \frac{\partial z^I(t; z_0)}{\partial z_0^J} = \{z^I(t, z_0), z_0^K\}_{z_0}(\pi^{-1})^{KJ}$$

then canonically quantize

$$\{z^I(t, z_0), z_0^K\}_{z_0} \mapsto \frac{i}{\hbar}[\hat{z}^I(t), \hat{z}^K(0)] = \frac{i}{\hbar}\begin{pmatrix}[\hat{x}(t), \hat{x}(0)] & [\hat{x}(t), \hat{p}(0)]\\ [\hat{p}(t), \hat{x}(0)] & [\hat{p}(t), \hat{p}(0)]\end{pmatrix}$$

and average over $\hat{\rho}$.

In what order?

- After extracting σ_{max}

$$\kappa = \lim_{t\to\infty}\frac{1}{t}\langle\log\frac{i}{\hbar}[\hat{x}(t), \hat{x}(0)]\rangle_\rho \text{ — not suitable for large } N$$

- Before extracting σ_{max}

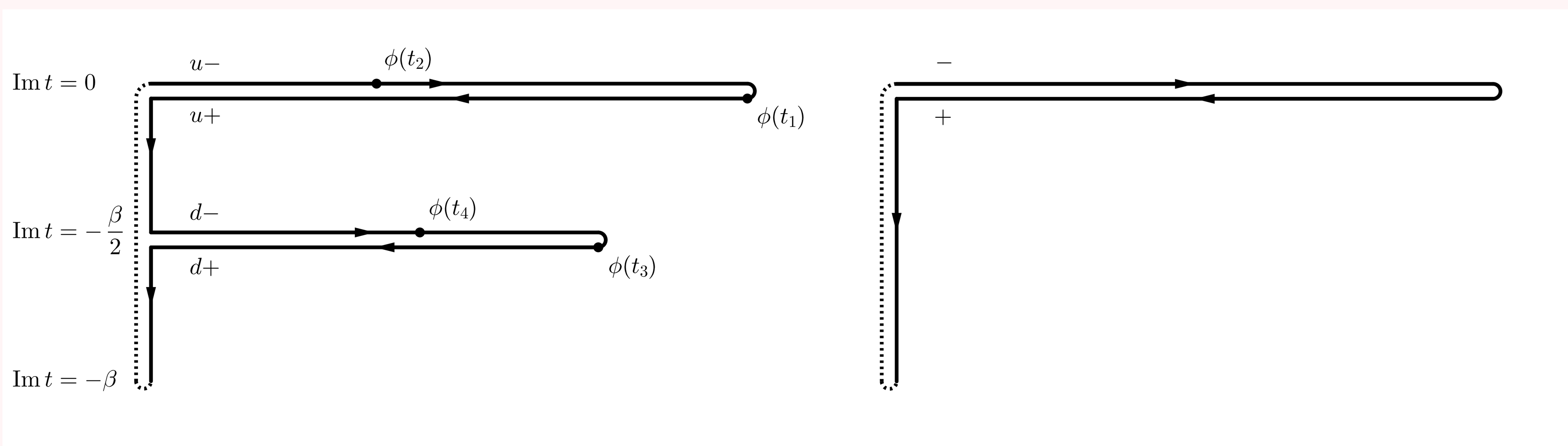
$$\langle\hat{\Phi}\rangle_\rho \sim \langle[\hat{x}_i(t_1), \hat{x}_j(t_2)]\rangle_\rho \text{ — vanishes or oscillates}$$

Right way (but not unique) — define **OTOC** $C_{ij}(t_1, t_2, t_3, t_4) \sim \langle\hat{\Phi}\hat{\Phi}^T\rangle_\rho$

$$\boxed{C_{ij}(t_1, t_2, t_3, t_4) = \mathrm{tr}\left(\hat{\rho}^{\frac{1}{2}}[\hat{x}_i(t_1), \hat{x}_j(t_2)]^\dagger\,\hat{\rho}^{\frac{1}{2}}[\hat{x}_i(t_3), \hat{x}_j(t_4)]\right)}$$

Some facts and issues on OTOCs

- OTOCs are very hard to calculate (doubled Schwinger-Keldysh diagram technique, etc).
- What is universal choice for $\hat{\rho}$?
- OTOCs and its classical counterpart are rarely compared in literature.



Main problem

Task is to calculate quantum and classical Lyapunov exponents for concrete model and compare results in different limits.

Our model

- Action — $O(N)$-model with diagonal terms excluded

$$S = \int dt\left[\sum_{i=1}^{N}\left(\frac{1}{2}\dot{x}_i^2 - \frac{m^2}{2}x_i^2\right) - \underbrace{\frac{\lambda}{4N}\sum_{i,j=1}^{N}x_i^2x_j^2}_{\text{symmetric}} + \underbrace{\frac{\lambda}{4N}\sum_{i=1}^{N}x_i^4}_{\text{nonsymmetric}}\right]$$

- Origins — $SU(2)$ Yang-Mills ($N = 3$ case)

$$\partial_i A_i^a = 0, \quad A_0^a = 0, \quad A_i^a \sim O_i^a x^a(t) \quad \to \quad \ddot{x}^a + [x^2 - (x^a)^2]x^a = 0$$

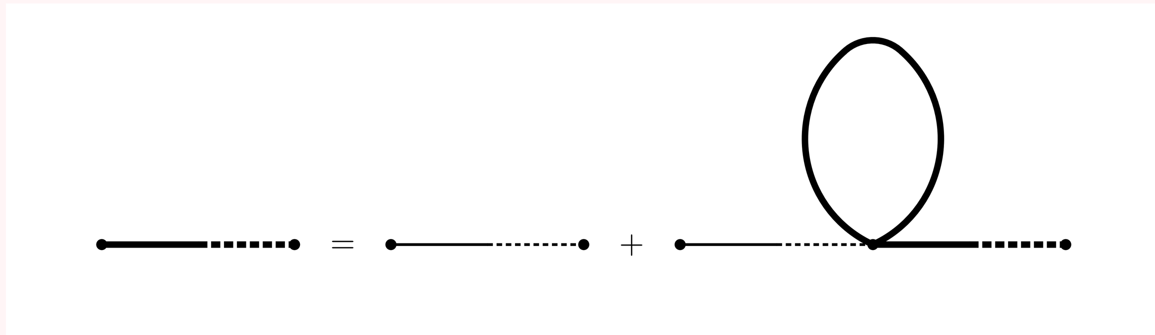
- $N = 2$ case — well studied

$$S_{N=2} = \int dt\left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{m^2}{2}x^2 - \frac{m^2}{2}y^2 - \frac{\lambda}{4}x^2y^2\right]$$

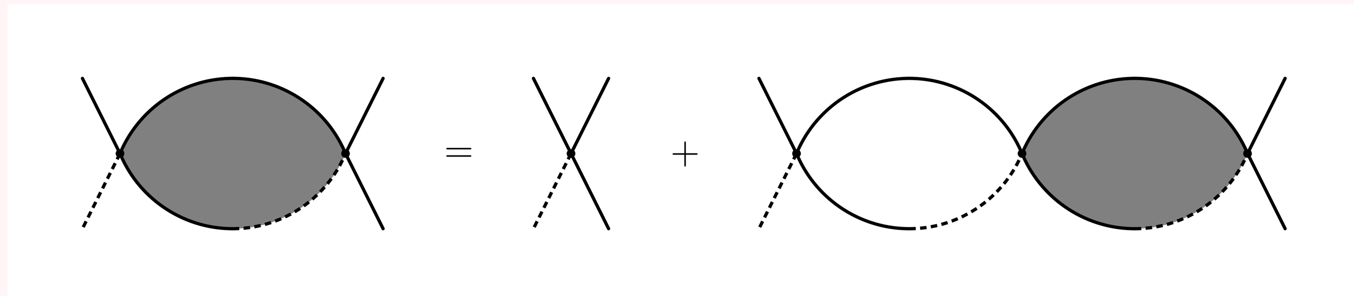
OTOC calculation

Strategy of OTOC calculation is as follows

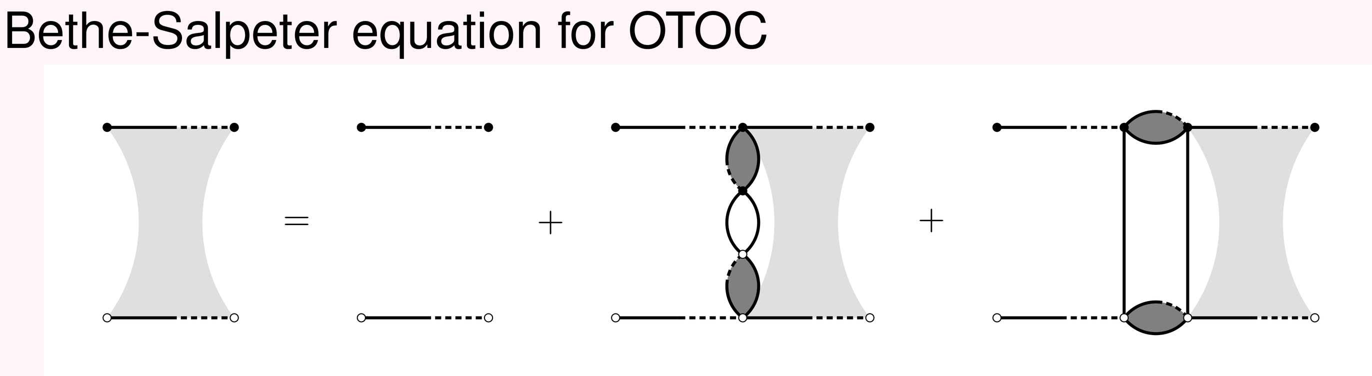
1. Resum self-energy diagrams



2. Resum bubble diagrams



3. Solve Bethe-Salpeter equation for OTOC



Final result is

$$\boxed{\kappa_q \approx \frac{8\sqrt{6}}{N}\left[6 - 2\frac{m^2}{\tilde{m}^2}\right]^{-\frac{3}{2}}\frac{e^{\beta\tilde{m}/2}}{e^{\beta\tilde{m}} - 1}\frac{\lambda}{\tilde{m}^2}}, \qquad \frac{\tilde{m}^2}{m^2} = 1 + \frac{\lambda}{2m^3}\frac{m}{\tilde{m}}\coth\left(\frac{\beta\tilde{m}}{2}\right),$$

and in high-temperature limit $\beta m \ll 1$, $\beta m \ll \lambda/m^3$ becomes

$$\kappa_q^{\text{high}} \approx \frac{4}{3N}\sqrt[4]{\frac{\lambda}{\beta}}\left(1 + \mathcal{O}\left(1/N, (\beta m^4/\lambda)^{\frac{1}{2}}\right)\right).$$

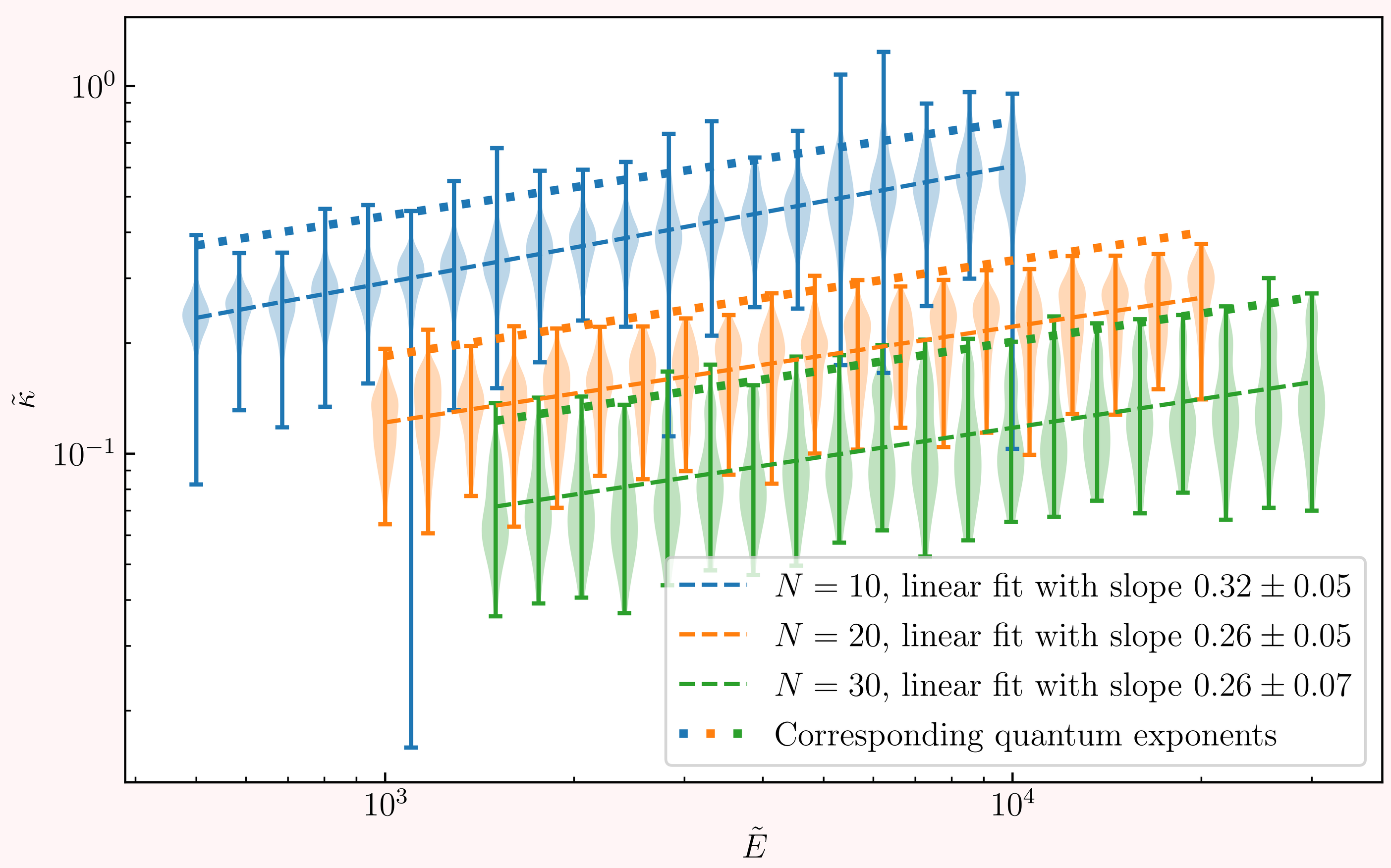
First quantum correction $\sim \hbar^2$ arises in third order of $\sqrt{\beta m^4/\lambda}$ expansion.

Numerical results

How to compare obtained result to classical Lyapynov exponent?

- Generate large sample of initial conditions for fixed d.o.f. number N and energy E.
- Solve ODE for z and Φ and calculate Lyapunov exponents.
- Repeat for different N and/or E.
- Average and fit the results.

For example, **energy dependence of classical Lyapunov exponent** show good coincidence to quantum counterpart



N. Kolganov and D. A. Trunin, "Classical and quantum butterfly effect in nonlinear vector mechanics," Phys. Rev. D **106** (2022), 025003, arXiv:2205.05663

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