Moscow International School of Physics 2022
Canonical description for formulation of embedding gravity as General Relativity with additional matter

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## Embedding theory and Regge-Teitelboim equations

Embedding theory: approach to describe gravity.


- embedding function $y^{a}\left(x^{\mu}\right): \mathcal{M} \rightarrow \mathbb{R}^{N_{+}, N_{-}}$
- $\mu=0,1,2,3$ $a=0,1, \ldots, N-1$
- ambient space metric $\eta_{a b}$ is flat
- surface metric $g_{\mu \nu}$ is induced: $g_{\mu \nu}=\left(\partial_{\mu} y^{a}\right)\left(\partial_{\nu} y^{b}\right) \eta_{a b}$

Einstein-Hilbert action: $S=\int d^{4} x \sqrt{-g}\left(\frac{R}{2 \varkappa}+\mathcal{L}_{m}\right)$
Indenendent variable: $g_{\mu}$ (OTO) $\rightarrow y^{a}(x)$ (embedding theory)
Equations of motion (Regge-Teitelboim equations)


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Einstein-Hilbert action: $S=\int d^{4} x \sqrt{-g}\left(\frac{R}{2 x}+\mathcal{L}_{m}\right)$
Independent variable: $g_{\mu \nu}(\mathrm{OTO}) \rightarrow y^{a}(x)$ (embedding theory)
Equations of motion (Regge-Teitelboim equations): $D_{\mu}\left(\left(G^{\mu \nu}-\varkappa T^{\mu \nu}\right) \partial_{\nu} y^{a}\right)=0$

## Embedding matter

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D_{\mu}\left(\left(G^{\mu \nu}-\varkappa T^{\mu \nu}\right) \partial_{\nu} y^{\mathrm{a}}\right)=0
$$

Regge-Teitelboim equations

- Einstein equations with the contribution of some additional (fictitious) matter with the energy-momentum tensor $\tau^{\mu \nu}$

- The embedding matter equation of motion:



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$$
G^{\mu \nu}-\varkappa\left(T^{\mu \nu}+\tau^{\mu \nu}\right)=0 ;
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Reformulating the embedding theory as GR with embedding matter at the level of action

The equivalence between embedding theory and GR with fictitious embedding matter at the level of equations of motion:

$$
D_{\mu}\left(\left(G^{\mu \nu}-\varkappa T^{\mu \nu}\right) \partial_{\nu} y^{a}\right)=0 \Leftrightarrow \begin{aligned}
& G^{\mu \nu}-\varkappa\left(T^{\mu \nu}+\tau^{\mu \nu}\right)=0 \\
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## The same equivalence at the level of action:



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\end{aligned}
$$

The same equivalence at the level of action:

$$
S\left[y^{a}\right]=\frac{1}{2 \varkappa} \int d^{4} x \sqrt{-g} R \Leftrightarrow \begin{aligned}
& S=S^{\text {EH }}+S_{\mathrm{m}}+S^{\text {add }} \\
& S^{\text {add }}=?
\end{aligned}
$$



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& S^{\text {add }}=?
\end{aligned}
$$

$$
S^{\text {add }}=\frac{1}{2} \int d^{4} x \sqrt{-g}\left(g_{\mu \nu}-\left(\partial_{\mu} y^{a}\right)\left(\partial_{\nu} y_{a}\right)\right) \tau^{\mu \nu}
$$

## Change of variables

Arnowitt-Deser-Mizner variables (ADM):

$$
\beta_{i k}=g_{i k}, \quad N_{k}=g_{0 k}, \quad N=\frac{1}{\sqrt{-g^{00}}} .
$$

The ADM action:


Notation:


## Change of variables

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\beta_{i k}=g_{i k}, \quad N_{k}=g_{0 k}, \quad N=\frac{1}{\sqrt{-g^{00}}} .
$$

The ADM action:

$$
S^{\mathrm{ADM}}=\int d^{4} \times\left(2 N K_{i k} L^{i k, l m} K_{l m}+\frac{1}{2 \varkappa} N \sqrt{\beta} R\right) .
$$

Notation:

$$
\begin{aligned}
& K_{i k}=\frac{1}{2 N}\left({ }^{3} D_{i} N_{k}+{ }_{D}^{3} N_{i}-\partial_{0} \beta_{i k}\right) \\
& L^{i k, l m}=\frac{\sqrt{\beta}}{8 \varkappa}\left(\beta^{i l} \beta^{k m}+\beta^{i m} \beta^{k l}-2 \beta^{i k} \beta^{l m}\right) \\
& \bar{L}_{i k, l m}=\frac{2 \varkappa}{\sqrt{\beta}}\left(\beta_{i l} \beta_{k m}+\beta_{i m} \beta_{k l}-\beta_{i k} \beta_{l m}\right)
\end{aligned}
$$

## Change of variables

From $\tau^{\mu \nu}$ to $\phi, \phi^{k}, \phi^{i j}$ :

$$
\begin{aligned}
& \phi=-\frac{1}{2} N^{2} \sqrt{\beta} \tau^{00} \\
& \phi^{k}=-N \sqrt{\beta} \tau^{k 0}-\frac{\phi}{N}\left(N^{k}+\beta^{i k} e_{i}^{b} e_{b 0}\right) \\
& \phi^{i j}=-\frac{1}{2} N \sqrt{\beta} \tau^{i j}+\beta^{i k} \bar{\beta}^{j m} \frac{\phi}{N} e_{k}^{a} e_{m}^{b} e_{0 a} e_{0 b}
\end{aligned}
$$

Notation:

$$
\begin{aligned}
& e_{\mu}^{a}=\partial_{\mu} y^{a} \\
& \bar{\beta}_{i k}=e_{i}^{a} e_{a k}
\end{aligned}
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From $\tau^{\mu \nu}$ to $\phi, \phi^{k}, \phi^{i j}$ :

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& \phi^{i j}=-\frac{1}{2} N \sqrt{\beta} \tau^{i j}+\beta^{i k} \bar{\beta}^{j m} \frac{\phi}{N} e_{k}^{a} e_{m}^{b} e_{0 a} e_{0 b}
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Notation:

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\begin{aligned}
& e_{\mu}^{a}=\partial_{\mu} y^{a} \\
& \bar{\beta}_{i k}=e_{i}^{a} e_{a k}
\end{aligned}
$$

$$
S^{\text {add }}=\int d^{4} x\left(\left(\bar{\beta}_{i j}-\beta_{i j}\right) \phi^{i j}+\left(e_{i}^{a} e_{a 0}-N_{i}\right) \phi^{i}+\left(N+\frac{1}{N} e_{a 0} \Pi_{\perp}^{a b} e_{b 0}\right) \phi\right)
$$

Here $\Pi_{\perp}^{3}{ }_{b}^{a}=\delta_{b}^{a}-e_{i}^{a} e_{b j} \bar{\beta}^{i j}$.

Momenta, primary constraints

Momenta corresponding to the ADM variables $\beta_{i k}, N_{k}, N$ :

$$
\begin{gathered}
\pi^{i k}=\frac{\delta S}{\delta \dot{\beta}_{i k}}=-2 L^{i k, l m} K_{l m}, \quad \pi_{N}^{k}=0, \quad \pi_{N}=0 . \\
\Phi=\pi_{N} \approx 0 \quad \\
\Phi^{k}=\pi_{N}^{k} \approx 0
\end{gathered}
$$

Momenta corresponding to $\phi, \phi^{i}, \phi^{i k}$ :

$$
\psi=\pi^{\phi} \approx 0 \quad \psi_{i}=\pi_{i}^{\phi} \approx 0 \quad \psi_{i k}=\pi_{i k}^{\phi} \approx 0
$$

Momenta corresponding to $y^{a}$ :

$$
\begin{gathered}
p_{a}=\frac{\delta S}{\delta \dot{y}^{a}}=\phi^{k} e_{a k}+\frac{2 \phi}{N} \prod_{\perp}^{3}{ }_{a}^{b} \dot{y}_{b} . \\
\Omega_{j}=p_{a} e_{j}^{a}-\phi^{k} \bar{\beta}_{k j} \approx 0
\end{gathered}
$$

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## Momenta corresponding to $\phi, \phi^{i}, \phi^{i k}$ :



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Momenta corresponding to $\phi, \phi^{i}, \phi^{i k}$ :

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\Psi=\pi^{\phi} \approx 0 \quad \Psi_{i}=\pi_{i}^{\phi} \approx 0 \quad \Psi_{i k}=\pi_{i k}^{\phi} \approx 0
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\end{gathered}
$$

## Hamiltonian

$$
\mathcal{H}=\pi^{i k} \dot{\beta}_{i k}+p_{a} \dot{y}^{a}-\mathcal{L}+\text { primary constraints with Lagrange multipliers }
$$

Hamiltonian density $\mathcal{H}$ can be conveniently broken down into two terms

$$
\mathcal{H}=\mathcal{H}^{\mathrm{ADM}}+\mathfrak{H}^{\text {add }}
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where


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\mathcal{H}^{\mathrm{ADM}}=\frac{1}{2} N \pi^{i k} \bar{L}_{i k, l m} \pi^{l m}+\pi^{i k}\left(\stackrel{3}{D}_{i} N_{k}+\stackrel{3}{D}_{k} N_{i}\right)-\frac{1}{2 \varkappa} N \sqrt{\beta} R{ }^{3}+\lambda \Phi+\lambda_{k} \Phi^{k}
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\mathcal{H}^{\text {add }}=\frac{N}{4 \phi} p_{a} \Pi_{\perp}^{3} p_{b}+\phi^{i j}\left(\beta_{i j}-\bar{\beta}_{i j}\right)+p_{a} e_{i}^{a} \bar{\beta}^{i k} N_{k}-\phi N+\chi \Psi+\chi^{i} \Psi_{i}+\chi^{i j} \Psi_{i j}+\xi^{i} \Omega_{i} .
\end{gathered}
$$

All the primary constraints $T_{\alpha}, \alpha=1 \ldots M$ must be preserved: $\dot{T}_{\alpha}=\left\{H, T_{\alpha}\right\} \approx 0$.
Conditions for the conservation of primary
constraints give the first generation of secondary constraints:


Next generation of secondary constraints:


## Secondary constraints

All the primary constraints $T_{\alpha}, \alpha=1 \ldots M$ must be preserved: $\dot{T}_{\alpha}=\left\{H, T_{\alpha}\right\} \approx 0$.
Conditions for the conservation of primary constraints give the first generation of secondary constraints:

$$
\begin{aligned}
& \bar{\equiv}=\phi+\frac{\zeta}{2} p_{\perp} \approx 0 ; \\
& \mathcal{H}_{0}=\mathcal{H}_{0}^{\mathrm{ADM}}+\zeta p_{\perp} \approx 0 ; \\
& \mathcal{H}_{k}=\mathcal{H}_{k}^{\mathrm{ADM}}+e_{k}^{a} p_{a} \approx 0 ; \\
& \Sigma_{i j}=\beta_{i j}-\bar{\beta}_{i j} \approx 0 .
\end{aligned}
$$

## Notation:

$$
\begin{aligned}
& \mathcal{H}_{0}^{\mathrm{ADM}}=\frac{1}{2} \pi^{i k} \bar{L}_{i k, l m} \pi^{\prime m}-\frac{1}{2 \varkappa} \sqrt{\beta}{ }^{3} ; \\
& \mathcal{H}_{i}^{\mathrm{ADM}}=-2 \beta_{i m} \sqrt{\beta}{ }^{3} \bar{D}_{j} \frac{\pi^{j m}}{\sqrt{\beta}} ; \\
& p_{\perp}=\sqrt{-p_{a} \Pi_{\perp}^{a b} p_{b}} ; \\
& \zeta= \pm 1 .
\end{aligned}
$$

Next generation of secondary constraints:


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& \Sigma_{i j}=\beta_{i j}-\bar{\beta}_{i j} \approx 0 .
\end{aligned}
$$

Next generation of secondary constraints:

$$
\begin{gathered}
\Lambda_{i k}=\bar{L}_{i k, l m} \pi^{l m}-2 \zeta n_{a}^{3} b_{i k}^{a} \approx 0 \\
\Upsilon_{i j}=\bar{\Upsilon}_{i j}+\phi^{k m} \Upsilon_{i j, k m}^{(1)} \approx 0
\end{gathered}
$$

## Notation:

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\begin{aligned}
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& \mathcal{H}_{i}^{\mathrm{ADM}}=-2 \beta_{i m} \sqrt{\beta}{ }^{3}{ }^{3} \frac{\pi^{j m}}{\sqrt{\beta}} \\
& p_{\perp}=\sqrt{-p_{a} \Pi_{\perp}^{3} p_{b}} ; \\
& \zeta= \pm 1 \\
& p_{\perp}^{a}=\Pi_{\perp}^{3 a b} p_{b} ; \\
& n^{a}=\frac{p_{\perp}^{a}}{p_{\perp}} \\
& b_{i k}^{a}=D_{i}^{3} D_{k}^{3} y^{a} .
\end{aligned}
$$

It is convenient to write the Hamiltonian density as a linear combination of the constraints already introduced:

$$
\mathcal{H}=N \mathcal{H}_{0}+N^{k} \mathcal{H}_{k}+\Sigma_{i j}\left(\phi^{i j}+\beta^{i k} \bar{\beta}^{j m} e_{k}^{a} p_{a} N_{m}\right)+\lambda \Phi+\lambda_{k} \Phi^{k}+\chi \Psi+\chi^{i} \Psi_{i}+\chi^{i j} \Psi_{i j}+\xi^{i} \Omega_{i} .
$$

## First and second class constraints

## First class constraints

First class constraints are constraints whose Poisson brackets with all other constraints are just constraints linear combinations:

$$
\left\{T_{\alpha}, T_{\beta}\right\}=C_{\alpha \beta}^{\gamma} T_{\gamma} \approx 0
$$

Quantizing the theory with first class constraints $T_{\alpha}, \alpha=1 \ldots M$ :

$$
\begin{gathered}
{\left[\widehat{T}_{\alpha}, \widehat{T}_{\beta}\right]=\widehat{C}_{\alpha \beta}^{\gamma} \widehat{T}_{\gamma} ;} \\
\widehat{T}_{\alpha}|\varphi\rangle=0 .
\end{gathered}
$$

This does not work with second class constraints!

## Constraints classification

- First class constraints: $\mathcal{H}_{k}$.
- Others are second class.


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## Constraints classification

- First class constraints: $\mathcal{H}_{k}$.
- Others are second class.

We have to solve second class constraints before quantization.

After solving the trivial constraints:
list of the constraints:

$$
\begin{aligned}
& \mathcal{H}_{0}=\mathcal{H}_{0}^{\mathrm{ADM}}+\zeta p_{\perp} \approx 0 \\
& \mathcal{H}_{k}=\mathcal{H}_{k}^{\mathrm{ADM}}+e_{k}^{a} p_{a} \approx 0 \\
& \Sigma_{i j}=\beta_{i j}-\bar{\beta}_{i j} \approx 0 \\
& \Lambda_{i k}=\bar{L}_{i k, l m} \pi^{l m}-2 \zeta n_{a}^{3} b_{i k}^{a} \approx 0
\end{aligned}
$$

variables:
$\begin{array}{ll}y^{a} & p_{a} \\ \beta_{i k} & \pi^{i k}\end{array}$

## Hamiltonian:

$\mathcal{H}=\widetilde{N} \mathcal{H}_{0}+\widetilde{N}^{k} \mathcal{H}_{k}+\Sigma_{i j}\left(\phi^{i j}+\beta^{i k} \bar{\beta}^{j m} e_{k}^{a} p_{a} N_{m}\right)$
Two options:

- Eliminate the embedding function $y^{a}$ and its conjugate momentum pa.
- Eliminate $\beta_{i j}$ and $\pi^{i j}$.


## After solving the trivial constraints:

## list of the constraints:

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\begin{aligned}
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& \Lambda_{i k}=\bar{L}_{i k, l m} \pi^{\prime m}-2 \zeta n_{a}^{3} b_{i k}^{a} \approx 0
\end{aligned}
$$

## variables:

$$
\begin{array}{ll}
y^{a} & p_{a} \\
\beta_{i k} & \pi^{i k}
\end{array}
$$

Hamiltonian:

$$
\mathcal{H}=\widetilde{N} \mathcal{H}_{0}+\widetilde{N}^{k} \mathcal{H}_{k}+\Sigma_{i j}\left(\phi^{i j}+\beta^{i k} \bar{\beta}^{j m} e_{k}^{a} p_{a} N_{m}\right) .
$$

## Two options

- Eliminate the embedding function $y^{a}$ and its conjugate momentum $p_{a}$
- Eliminate $\beta_{i j}$ and $\pi^{i j}$


## list of the constraints:

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\begin{aligned}
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& \Lambda_{i k}=\bar{L}_{i k, l m} \pi^{l m}-2 \zeta n_{a} b_{i k}^{3} \approx 0
\end{aligned}
$$

## variables:

$$
\begin{array}{ll}
y^{a} & p_{a} \\
\beta_{i k} & \pi^{i k}
\end{array}
$$

Hamiltonian:

$$
\mathcal{H}=\widetilde{N} \mathcal{H}_{0}+\widetilde{N}^{k} \mathcal{H}_{k}+\Sigma_{i j}\left(\phi^{i j}+\beta^{i k} \bar{\beta}^{j m} e_{k}^{a} p_{a} N_{m}\right) .
$$

Two options:

- Eliminate the embedding function $y^{a}$ and its conjugate momentum $p_{a}$.
- Eliminate $\beta_{i j}$ and $\pi^{i j}$.


## list of the constraints:

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- Eliminate $\beta_{i j}$ and $\pi^{i j}$. $\longleftarrow$ we choose this way

Solving the remaining second class constraints

Solution:

$$
\beta_{i j}=\bar{\beta}_{i j} ; \quad \pi^{i j}=2 \zeta n_{a} b_{l k}^{a} L^{k k, i j} .
$$

## First order action:

$$
S^{(1)}=\int d t \int d^{3} x\left(\pi^{i k} \dot{\beta}_{i k}+p_{a} \dot{y}^{a}-N \mathcal{H}_{0}-N^{k} \mathcal{H}_{k}\right)
$$

Then

here $B^{a b}=4 b_{i k}^{a} b_{l m}^{b} L^{i k, l m}$

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Then

$$
\mathcal{L}^{(1)}=-\zeta\left(B_{a b} n^{b}+\frac{1}{2}\left(n_{c} B^{c b} n_{b}+B_{c}^{c}\right) n_{a}\right) \dot{y}^{a}-N \mathcal{H}_{0}-N^{k} \mathcal{H}_{k}
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$$

## Primary constraints:



Secondary constraints:

$\mathcal{H}_{k}=\mathcal{H}_{k}^{\mathrm{ADM}}\left(y^{a}, \pi_{a}\right)+e_{k}^{a} p_{a} \approx 0 ;$
$N \approx 0$;
$N_{k} \approx 0$

$$
\mathcal{L}^{(1)}=-\zeta\left(B_{a b} n^{b}+\frac{1}{2}\left(n_{c} B^{c b} n_{b}+B_{c}^{c}\right) n_{a}\right) \dot{y}^{a}-N \mathcal{H}_{0}-N^{k} \mathcal{H}_{k}
$$

Primary constraints:

## Secondary constraints:

$$
\begin{aligned}
& \widehat{\Phi}_{i}=\pi_{a} e_{i}^{a} \approx 0 ; \\
& \widehat{\Phi}_{4}=n\left(y^{a}, \pi_{a}\right)^{2}+1 \approx 0 ; \\
& \widehat{\Psi}^{a}=\frac{n^{a}\left(y^{a}, \pi_{a}\right)}{\sqrt{-n^{2}}}-\frac{p_{\perp}^{a}}{p_{\perp}} \approx 0 ; \\
& \pi_{p}^{a} \approx 0 ; \\
& \pi_{N} \approx 0 ; \\
& \pi_{N}^{k} \approx 0 .
\end{aligned}
$$

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& N \approx 0 ; \\
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\end{aligned}
$$

Hamiltonian after solving the constraints:

$$
\mathcal{H}=\widehat{\chi}^{i} \widehat{\Phi}_{i}+\widehat{\chi} \widehat{\Phi}_{4} .
$$

## Conclusions

For the complete embedding theory formulated in the form of GR with an additional contribution of the so-called embedding matter, the canonical description of the theory is constructed.

- All the constraints are found.
- Some of the constraints are second class.
- Second class constraints are solved by eliminating the variables $\beta_{i k}, \pi^{i k}$
- The constructed canonical system passes into the canonical formulation of the complete embedding theory.


## Next steps

- Solving the constraints by excluding canonical variables in such a way that the variables $\beta_{i k}$, $\pi^{i k}$, as well as variables describing embedding matter remain
- Studying the Einstein limit.


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