# Generalized method of symmetric embeddings construction and its application to spacetimes of general relativity

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# Outline

- Isometric embeddings
- Two-dimensional sphere example
- Bendable embeddings building
- (2 + 1)-dimensional gravity examples
- Rotaitng BTZ example
- Conclusions

# Isometric embeddings

Isometrically embedding a (pseudo)Riemannian manifold in a flat space means finding a surface in a flat space of a larger number of dimensions on which the induced metric will coincide with the metric of the original one.

# Isometric embeddings

Isometrically embedding a (pseudo)Riemannian manifold in a flat space means finding a surface in a flat space of a larger number of dimensions on which the induced metric will coincide with the metric of the original one.
However, even if an embedding in a flat space is found, it is not necessarily unique. In the case when the embeddings pass into each other with a continuous change of some parameter, then we are talking about the possibility of isometric bending of the embedding.

# Two dimensional sphere example

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2. \tag{1}$$

2D sphere in $\mathbb{R}^3$:

$$\begin{aligned} y^1 &= \sin\theta\cos\varphi, \\ y^2 &= \sin\theta\sin\varphi, \\ y^3 &= \cos\theta. \end{aligned} \tag{2}$$

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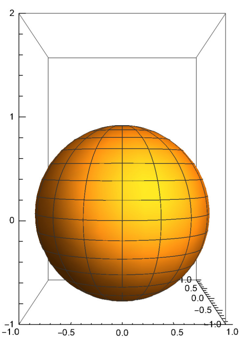
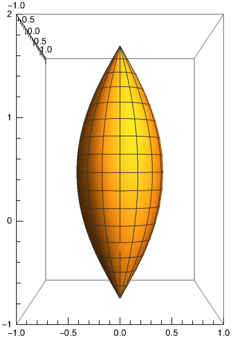
Bended sphere:

$$\begin{aligned} y^1 &= \frac{1}{a}\sin\theta\cos a\varphi, \\ y^2 &= \frac{1}{a}\sin\theta\sin a\varphi, \\ y^3 &= \int_0^\theta \sqrt{1 - \frac{\cos^2\theta'}{a^2}} d\theta'. \end{aligned} \tag{3}$$

# Two-dimensional sphere example




$a = 1$




$a = 2$

# Building bendable embeddings

$$ds^2 = g_{ij}(x_1, ..., x_n)dx^i dx^j + F^2(x_1, ..., x_n)d\varphi^2 \tag{4}$$

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$$\begin{aligned} y^1 &= \sqrt{\xi}\frac{F}{a}\sin\sqrt{\xi}a\varphi, \\ y^2 &= \frac{F}{a}\cos\sqrt{\xi}a\varphi. \end{aligned} \tag{5}$$

$$\tilde{ds}^2 = g_{ij}(x_1, ..., x_n)dx^i dx^j - \xi\frac{dF(x_1, ..., x_n)^2}{a^2} \tag{6}$$

# Building bendable embeddings

$$ds^2 = g_{ij}(x_1, ..., x_n)dx^i dx^j + F^2(x_1, ..., x_n)d\varphi^2 + G(x_1, ..., x_n)dx^i d\varphi \quad (7)$$

# Building bendable embeddings

$$ds^2 = g_{ij}(x_1, ..., x_n)dx^i dx^j + F^2(x_1, ..., x_n)d\varphi^2 + G(x_1, ..., x_n)dx^i d\varphi \tag{7}$$

$$\begin{aligned} y^1 &= \sqrt{\xi}\sqrt{\frac{G(x_1, ..., x_n)}{2bc}}\sin(\sqrt{\xi}(bx^i - c\varphi)), \\ y^2 &= \sqrt{\frac{G(x_1, ..., x_n)}{2bc}}\cos(\sqrt{\xi}(bx^i - c\varphi)), \end{aligned} \tag{8}$$

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$$ds^2 = \tilde{g}_{ij}(x_1, ..., x_n)dx^i dx^j + (F^2(x_1, ..., x_n) + \eta G(x1, ..., x_n)\frac{c}{2b})d\varphi^2 \tag{9}$$

# (2 + 1)-dimensional gravity examples

$\Lambda > 0$[1]:

$$ds^2 = \cos^2\theta dt^2 - \frac{1}{\Lambda}(d\theta^2 + \sin^2\theta d\varphi^2), \tag{10}$$

где $t \subset (-\infty, \infty)$, $\theta \subset [0, \pi)$, $\varphi \subset [0, 2\pi(1 - 4GM))$.

[1] S. Desert and R. Jekyw. Three-dimensional cosmological gravity: dynamics of constant curvature. Annals of Physics, 1984

# (2 + 1)-dimensional gravity examples

$$
\begin{aligned}
y^1 &= \frac{1}{a\sqrt{\Lambda}}\cos\theta\sinh a\sqrt{\Lambda}t, \\
y^2 &= \frac{1}{a\sqrt{\Lambda}}\cos\theta\cosh a\sqrt{\Lambda}t, \\
y^3 &= \frac{1}{b\sqrt{\Lambda}}\sin\theta\sin b\varphi, \\
y^4 &= \frac{1}{b\sqrt{\Lambda}}\sin\theta\cos b\varphi. \\
y^5 &= \frac{1}{\sqrt{\Lambda}}\int_0^\theta \sqrt{1-\frac{\sin^2(\theta')}{a^2}-\frac{\cos^2(\theta')}{b^2}}d\theta'
\end{aligned} \qquad (11)
$$

, where $\varphi \subset [0, \frac{2\pi}{b})$, $b = 1/(1-4GM) \geq 1$, $a \geq 1$.

# (2 + 1)-dimensional gravity examples

Let $a = b$

$$\Rightarrow y^5 = \frac{1}{\sqrt{\Lambda}}\sqrt{1 - \frac{1}{b^2}}\theta \tag{12}$$

# (2 + 1)-dimensional gravity examples

Magnetic monopole in (2+1) dimensions[2], $\Lambda = -1/l^2$:

$$ds^2 = \frac{1}{l^2}(x^2 + \bar{r}^2 - r_+^2)dt^2 - \frac{l^2 x^2 dx^2}{\left(x^2 + \bar{r}^2 - r_+^2\right)\left(x^2 + Q_m^2 \ln\left[1 + \frac{x^2}{\bar{r}^2 - r_+^2}\right]\right)} - \left(x^2 + Q_m^2 \ln\left[1 + \frac{x^2}{\bar{r}^2 - r_+^2}\right]\right) d\varphi^2 \quad (13)$$

where $Q_m$, $M$, $\bar{r}$ – some parameters, $r_+^2 = Ml^2$

$t \subset (0, \infty)$, $x \subset (0, \infty)$, $\varphi \subset (0, 2\pi T_\varphi]$ and $T_\varphi = \dfrac{\exp(\bar{r}^2/2Q_m^2)}{1 + Q_m^2 \exp(\bar{r}^2/2Q_m^2)}$

[2] Hirschmann E. W., Welch D. L. Magnetic solutions to 2+ 1 gravity //Physical Review D. – 1996. – T. 53. – №. 10. – C. 5579.

$$
\begin{aligned}
y^1 &= \frac{1}{al}\sqrt{x^2+\bar{r}^2-r_+^2}\,\sinh at,\\
y^2 &= \frac{1}{al}\sqrt{x^2+\bar{r}^2-r_+^2}\,\cosh at,\\
y^3 &= \frac{1}{b}\sqrt{x^2+Q_m^2\ln\left[1+\frac{x^2}{\bar{r}^2-r_+^2}\right]}\,\sin b\varphi\\
y^4 &= \frac{1}{b}\sqrt{x^2+Q_m^2\ln\left[1+\frac{x^2}{\bar{r}^2-r_+^2}\right]}\,\cos b\varphi\\
y^5 &= \int lx(x^2+\bar{r}^2-r_+^2)^{-1/2}(x^2+Q_m^2\ln[1+\frac{x^2}{\bar{r}^2-r_+^2}])^{-1/2}dx,\\
y^6 &= \int\frac{1}{abl}\frac{x}{f(x)}\sqrt{\frac{G(x)}{x^2+Q_m^2\ln[1+\frac{x^2}{\bar{r}^2-r_+^2}]}}dx,\\
G(x) &= a^2l^2f(x)^2+(b^2x^2+2Q_m^2a^2l^2+Q_m^2b^2\ln[1+\frac{x^2}{\bar{r}^2-r_+^2}])f(x)+Q_m^2l^2a^2,\\
f(x) &= x^2+\bar{r}^2-r_+^2
\end{aligned}
\tag{14}
$$

# Rotating BTZ example

Rotating BTZ in Eddington-Finkelstein coordinates[3]:

$$ds^2 = \left(-M + \frac{r^2}{l^2}\right) dv^2 - 2dvdr + Jdvd\bar{\varphi} - r^2 d\bar{\varphi}^2 \tag{15}$$

[3] Sheykin A. A., Markov M. V., Paston S. A. Global embedding of BTZ spacetime using generalized method of symmetric embeddings construction //Journal of Mathematical Physics. – 2021. – T. 62. – №. 10.

# Rotating BTZ example

$$
\begin{aligned}
y^1 &= \sqrt{\frac{1}{a}} \sin\left(\frac{aJ}{2} v + \bar{\varphi} - \frac{2}{J} r\right), \\
y^2 &= \sqrt{\frac{1}{a}} \cos\left(\frac{aJ}{2} v + \bar{\varphi} - \frac{2}{J} r\right), \\
y^3 &= \sqrt{r^2 + \frac{1}{a}} \cos\left(\bar{\varphi} - \frac{2}{J} \frac{\arctan(\sqrt{a} r)}{\sqrt{a}}\right), \\
y^4 &= \sqrt{r^2 + \frac{1}{a}} \sin\left(\bar{\varphi} - \frac{2}{J} \frac{\arctan(\sqrt{a} r)}{\sqrt{a}}\right), \\
y^5 &= bv, \\
y^6 &= \frac{1}{c} \sqrt{b^2 - M + \frac{r^2}{l^2} - \frac{aJ^2}{4}} \sin(cv), \\
y^7 &= \frac{1}{c} \sqrt{b^2 - M + \frac{r^2}{l^2} - \frac{aJ^2}{4}} \cos(cv),
\end{aligned} \tag{16}
$$

# Conclusions

- Has been developed a method that makes it possible to construct isometrically bendable embeddings of sufficiently symmetric spaces.
- It turns out that the resulting method is also useful for the problem of constructing explicit embeddings in general.
- The main result from the point of view of the problem of constructing explicit embeddings is the global embedding of a BTZ black hole into a 7-dimensional flat space.