Quantum chaos in nonlinear vector mechanics

based on arXiv:2205.05663

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July 28, 2022

Classical and quantum chaos

Classical chaos is closely related to the exponential sensitivity to initial conditions (“butterfly effect”):

$$k \;\mathbf{z}(t)k \quad e^{\;_{cl}t}k \;\mathbf{z}(0)k;$$

where $\;_{cl}$ is called the classical Lyapunov exponent

Quantum chaos and quantum Lyapunov exponent are more subtle because there are no trajectories in quantum world

Due to this reason, we need to find **alternative signatures of chaos** that are well defined in the quantum case and distinct chaotic and integrable systems in the semiclassical limit

One of such signatures, which has recently grown popular, is the exponential growth of the **out-of-time-ordered correlation functions (OTOCs)**:

$$C(t) = \frac{1}{N^2} \sum_{i,j=1}^{N} \left\langle [\hat{q}_i(t), \hat{p}_j(0)]^\dagger [\hat{q}_i(t), \hat{p}_j(0)] \right\rangle$$

In the semiclassical limit, OTOCs capture the "butterfly effect":

$$C(t) \approx \frac{1}{N^2} \sum_{i,j=1}^{N} \left\{ q_i(t), p_j(0) \right\}^2 = \frac{\hbar^2}{N^2} \sum_{i,j=1}^{N} \left| \frac{\partial q_i(t)}{\partial q_j(0)} \right|^2 \sim \hbar^2 \frac{\|\mathbf{z}(t)\|^2}{\|\mathbf{z}(0)\|^2} \sim \hbar^2 e^{2\lambda t}$$

OTOCs allow us to define the **quantum Lyapunov exponent**:

$$\lambda_q \equiv \frac{1}{2t} \log \left[\frac{1}{\hbar^2} \frac{1}{N^2} \sum_{i,j} C_{ij}(t) \right] \quad \text{as} \quad \frac{1}{\lambda_q} \ll t \ll \frac{1}{\lambda_q} \log \frac{1}{\hbar}$$

Note that eventually OTOCs are saturated, which reflects the breakdown of the semsiclassical description (cf. the Ehrenfest time)

Correspondence

Unfortunately, the correspondence between the classical and quantum chaos remains **relatively poorly studied**

Besides, there are few examples where OTOCs can be calculated analytically

Therefore, it is useful to consider a tractable model, where this **correspondence can be checked directly**

As an example of such a model, we propose the **vector mechanics** with a large number of degrees of freedom N and quartic interaction:

$$S = \int dt \sum_{i=1}^{N} \left[\frac{1}{2}\dot{\phi}_i^2 - \frac{m^2}{2}\phi_i^2 \right] - \underbrace{\frac{\lambda}{4N}\sum_{i,j=1}^{N}\phi_i^2\phi_j^2}_{\text{symmetric}} + \underbrace{\frac{(\ )^2}{4N}\sum_{i=1}^{N}\phi_i^4}_{\text{nonsymmetric}}$$

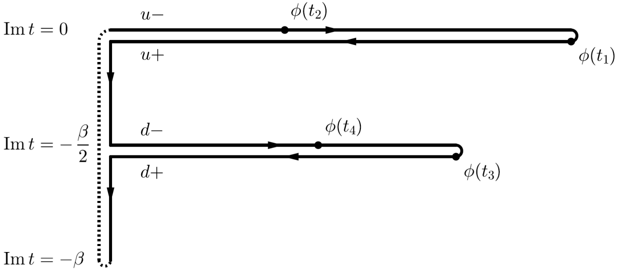
We also assume the system to be thermal with an inverse temperature

We will show that the symmetric model (= 0) is both classically and quantum integrable, whereas the nonsymmetric model (≠ 0) is chaotic

Augmented Schwinger-Keldysh technique

To calculate the regularized OTOC, we use the augmented Schwinger-Keldysh technique on the **twofold contour** (in our notation $C(t) = \partial_{t_1}\partial_{t_2} C_{12;34}\Big|_{t_3=t_4=0}^{t_1=t_2=t}$):

$$\begin{aligned} C_{12;34} = & -\langle \phi_{u+}(t_1)\,\phi_{u-}(t_3)\,\phi_{d+}(t_2)\,\phi_{d-}(t_4)\rangle - \langle \phi_{u-}(t_1)\,\phi_{u+}(t_3)\,\phi_{d-}(t_2)\,\phi_{d+}(t_4)\rangle \\ & + \langle \phi_{u+}(t_1)\,\phi_{u-}(t_3)\,\phi_{d-}(t_2)\,\phi_{d+}(t_4)\rangle + \langle \phi_{u-}(t_1)\,\phi_{u+}(t_3)\,\phi_{d+}(t_2)\,\phi_{d-}(t_4)\rangle \\ = & -\langle \phi_{uc}(t_1)\,\phi_{dc}(t_2)\,\phi_{uq}(t_3)\,\phi_{dq}(t_4)\rangle . \end{aligned}$$



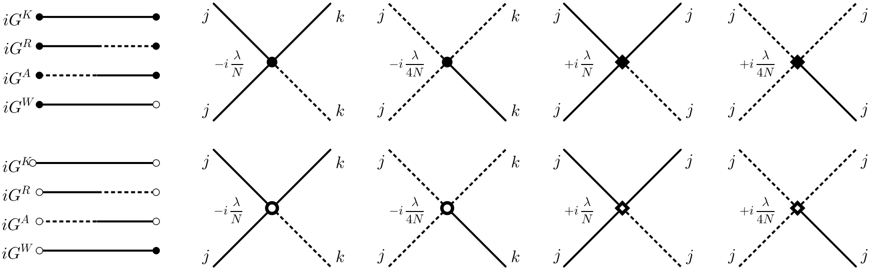
Augmented Schwinger-Keldysh technique

The **vertices are the same** as in the standard (onefold) technique

In addition to the standard R/A/K propagators, the augmented technique contains the **W propagator that connects different folds**:

$$iG_0^R(t_1;t_2) = \quad i \ \ (t_{12}) \frac{\sin(mt_{12})}{m}; \qquad iG_0^A(t_1;t_2) = i \ (\ \ t_{12}) \frac{\sin(mt_{12})}{m};$$

$$iG_0^K(t_1;t_2) = \frac{1}{2} \coth \frac{\ \ m}{2} \frac{\cos(mt_{12})}{m}; \qquad iG_0^W(t_1;t_2) = \frac{e^{\ \ m=2}}{e^{\ \ m} \quad 1} \frac{\cos(mt_{12})}{m}:$$



No chaos in the symmetric model

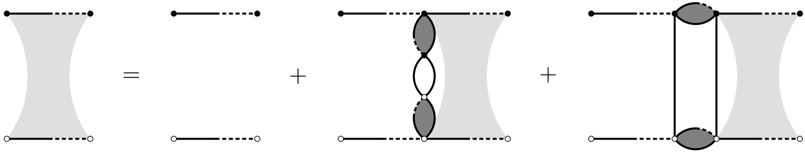
The leading corrections to the averaged OTOC in the $O(N)$-symmetric model are described by the so-called **“ladder” diagrams**

Substituting the **exponential ansatz** $C_{12;34} \quad e^{2 \quad t}$, $t = \frac{1}{2}(t_1 + t_2 \quad t_3 \quad t_4)$, into the Bethe-Salpeter equation, we get the equation on   :

$$1 \quad \frac{64}{N} \frac{w^2 \quad ^2}{m^6} \frac{1}{4} \frac{1}{\left(1 + \frac{\ ^2}{m^2}\right)^2} + \frac{4}{N} \frac{w^2 \quad ^2}{m^6} \frac{5 + \frac{\ ^2}{m^2}}{\left((\quad + 1)^2 + \frac{\ ^2}{m^2}\right)\left((\quad 1)^2 + \frac{\ ^2}{m^2}\right)\left(1 + \frac{\ ^2}{m^2}\right)};$$

where $w = e^{\ m=2} = \ e^{\ m} \quad 1$, m is the renormalized mass, and   is the parameter of the resummed vertices (shaded bubbles on the picture)

All solutions to this equation are purely inaginary; hence, there is **no quantum chaos in the $O(N)$-symmetric model**



Chaos in the full nonsymmetric model

Keeping in mind the leading contributions from nonsymmetric vertices and using the same ansatz for $C_{12;34}$, we get the equation on in the full model:

$$\frac{1}{\ 2} \quad \frac{1536}{N^2}\frac{w^2\ \ ^2}{\bar{m}^6}\frac{1}{\ 6}\frac{1}{\left(1+\frac{\ ^2}{\bar{m}^2}\right)^2} \quad \frac{24}{N^2}\frac{w^2\ \ ^2}{\bar{m}^6}\frac{5+\frac{\ ^2}{\bar{m}^2}\quad 3\ ^2\quad 3+(\ ^2+6)\ ^2+\ ^4}{1+\frac{\ ^2}{\bar{m}^2}\quad (\ +1)^2+\frac{\ ^2}{\bar{m}^2}\ ^2\quad (\ \ 1)^2+\frac{\ ^2}{\bar{m}^2}\ ^2}$$

The solutions to this equation **has a positive real part**

The maximal quantum Lyapunov exponent is as follows:

$$_q = \frac{8\sqrt{6}}{3}\frac{w}{\bar{m}^3}\frac{\bar{m}}{N}$$

The exponent scales as $_q \quad \sqrt[4]{\ \ =\ \ }$ in the high-temperature limit and is exponentially suppressed in the low-temperature limit:

$$^{\text{high}}_q \quad \frac{4}{3}\frac{\ }{N}\sqrt{\ ^4\ \frac{\ }{\ }}; \qquad ^{\text{low}}_q \quad \sqrt{6}\frac{\ }{N}\frac{\ }{m^3}m\exp\quad\frac{\ m}{2}$$

Comparison to classical chaos

The $O(N)$-**symmetric** model is clearly **classically integrable**, i.e., its maximal Lyapunov exponent is zero

The numerical calculations of the maximal Lyapunov exponent in the **nonsymmetric** model yield the following **high-temperature behavior**:

$$_{cl} \quad (1{:}3 \quad 0{:}2) \frac{1}{N^{1{:}18 \quad 0{:}05}} \quad - \quad {}^{0{:}28 \quad 0{:}02} \quad ;$$

where we assume $\quad = 1$ and use the relation $\quad N{=}E$

Details of this calculations are discussed in **Nikita Kolganov's poster**, which I kindly ask you to examine

Thus, in both models, classical and quantum Lyapunov exponents **approximately coincide with each other**

Qualitative analysis

In fact, the high-temperature behavior of classical and quantum Lyapunov exponents can be **deduced from dimensional grounds**

In the limit $m \quad 1$ and $m \quad /m^3$, the quadratic part of the potential energy is negligible, so the Hamiltonian acquires the following form ($\ = 1$):

$$H^{\text{high}} \quad \sum_{i=1}^{N} \frac{1}{2} \,{}_i^2 + \frac{}{4N} \sum_{i \neq j} \,{}_i^2 \,{}_j^2$$

This "pruned" Hamiltonian is **invariant under the scale transformations**:

$$t \to {}^{1}t; \qquad {}_i \to {}_i; \qquad H \to {}^{4}H$$

Since the Lyapunov exponent has the dimension of inverse time, this invariance implies the high-temperature dependence $\quad \sqrt[4]{E} \quad \sqrt[4]{\ =\ }$

- Furthermore, we can compare the **constant potential energy surface** (CPE surface) with a wall of a **Sinai billiard**
- It is known that Sinai billiards exhibit a chaotic behavior in the presence of **concave walls**
- In the nonsymmetric model, the CPE surface becomes concave at energies $E > E_{\mathrm{con}} = 3Nm^4{=}2$, which **agrees with the emergence of chaos**

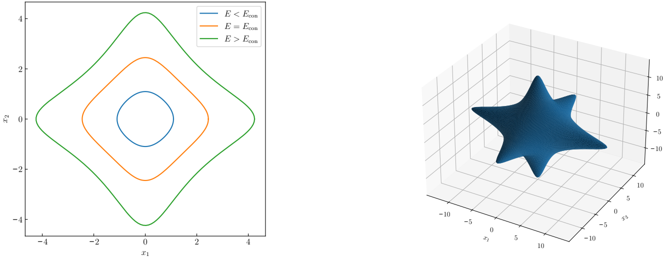


Figure: [Left] CPE curve for $N = 2$ and $E < E_{\mathrm{con}}$ (blue line), $E = E_{\mathrm{con}}$ (orange line), $E > E_{\mathrm{con}}$ (green line). [Right] CPE surface for $N = 3$ and $E \quad E_{\mathrm{con}}$.

Conclusion

- We suggest a tractable chaotic model — the nonlinear vector mechanics with a quartic interaction and thermal initial state
- In the $O(N)$-symmetric case, both classical and quantum Lyapunov exponents are zero
- In the nonsymmetric case, both exponents emerge in the high-temperature limit, approximately coincide, and scale as $_q \quad _{cl} \quad \frac{1}{N} \sqrt[4]{-}$
- This calculation supports the use of OTOCs as a diagnostic of quantum chaos