Quantum chaos in nonlinear vector mechanics

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Classical and quantum chaos

- **Classical chaos** is closely related to the exponential sensitivity to initial conditions ("butterfly effect"):

$$\|\delta \mathbf{z}(t)\| \sim e^{\kappa_{cl} t} \|\delta \mathbf{z}(0)\|,$$

where κ_{cl} is called the classical Lyapunov exponent

- **Quantum chaos** and quantum Lyapunov exponent are more subtle because there are no trajectories in quantum world

- Due to this reason, we need to find **alternative signatures of chaos** that are well defined in the quantum case and distinct chaotic and integrable systems in the semiclassical limit

OTOCs

- One of such signatures, which has recently grown popular, is the exponential growth of the **out-of-time-ordered correlation functions (OTOCs)**:

$$C(t) = \frac{1}{N^2} \sum_{i,j=1}^{N} \left\langle [\hat{q}_i(t), \hat{p}_j(0)]^{\dagger} [\hat{q}_i(t), \hat{p}_j(0)] \right\rangle$$

- In the semiclassical limit, OTOCs capture the "butterfly effect":

$$C(t) \approx \frac{1}{N^2} \sum_{i,j=1}^{N} \{q_i(t), p_j(0)\}^2 = \frac{\hbar^2}{N^2} \sum_{i,j=1}^{N} \left| \frac{\partial q_i(t)}{\partial q_j(0)} \right|^2 \sim \hbar^2 \frac{\|\mathsf{z}(t)\|^2}{\|\mathsf{z}(0)\|^2} \sim \hbar^2 e^{2\kappa t}$$

- OTOCs allow us to define the **quantum Lyapunov exponent**:

$$\kappa_q \approx \frac{1}{2t} \log \left[\frac{1}{\hbar^2} \frac{1}{N^2} \sum_{i,j} C_{ij}(t) \right] \quad \text{as} \quad \frac{1}{\kappa_q} \ll t \ll \frac{1}{\kappa_q} \log \frac{1}{\hbar}$$

- Note that eventually OTOCs are saturated, which reflects the breakdown of the semsiclassical description (cf. the Ehrenfest time)

Correspondence

- Unfortunately, the correspondence between the classical and quantum chaos remains **relatively poorly studied**
- Besides, there are few examples where OTOCs can be calculated analytically
- Therefore, it is useful to consider a tractable model, where this **correspondence can be checked directly**
- As an example of such a model, we propose the **vector mechanics** with a large number of degrees of freedom N and quartic interaction:

$$S = \int dt \left[\sum_{i=1}^{N} \left(\frac{1}{2}\dot{\phi}_i^2 - \frac{m^2}{2}\phi_i^2 \right) - \underbrace{\frac{\lambda}{4N} \sum_{i,j=1}^{N} \phi_i^2 \phi_j^2}_{\text{symmetric}} + \underbrace{\alpha^2 \frac{\lambda}{4N} \sum_{i=1}^{N} \phi_i^4}_{\text{nonsymmetric}} \right]$$

- We also assume the system to be thermal with an inverse temperature β
- We will show that the symmetric model ($\alpha = 0$) is both classically and quantum integrable, whereas the nonsymmetric model ($\alpha \neq 0$) is chaotic

Augmented Schwinger-Keldysh technique

To calculate the regularized OTOC, we use the augmented Schwinger-Keldysh technique on the **twofold contour** (in our notation $C(t) = \partial_{t_1}\partial_{t_2}C_{12;34}\big|_{t_3=t_4=0}^{t_1=t_2=t}$):

$$
\begin{aligned}
C_{12;34} = & -\langle \phi_{u+}(t_1)\phi_{u-}(t_3)\phi_{d+}(t_2)\phi_{d-}(t_4)\rangle - \langle \phi_{u-}(t_1)\phi_{u+}(t_3)\phi_{d-}(t_2)\phi_{d+}(t_4)\rangle \\
& + \langle \phi_{u+}(t_1)\phi_{u-}(t_3)\phi_{d-}(t_2)\phi_{d+}(t_4)\rangle + \langle \phi_{u-}(t_1)\phi_{u+}(t_3)\phi_{d+}(t_2)\phi_{d-}(t_4)\rangle \\
= & -\langle \phi_{uc}(t_1)\phi_{dc}(t_2)\phi_{uq}(t_3)\phi_{dq}(t_4)\rangle .
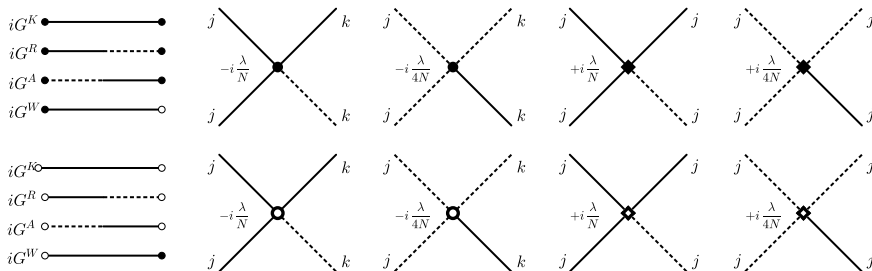\end{aligned}
$$
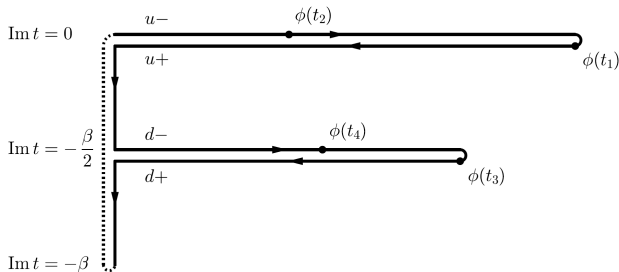
Augmented Schwinger-Keldysh technique

- The **vertices are the same** as in the standard (onefold) technique
- In addition to the standard R/A/K propagators, the augmented technique contains the **W propagator that connects different folds**:

$$iG_0^R(t_1, t_2) = -i\theta(t_{12})\frac{\sin(mt_{12})}{m}, \qquad iG_0^A(t_1, t_2) = i\theta(-t_{12})\frac{\sin(mt_{12})}{m},$$

$$iG_0^K(t_1, t_2) = \frac{1}{2}\coth\frac{\beta m}{2}\frac{\cos(mt_{12})}{m}, \qquad iG_0^W(t_1, t_2) = \frac{e^{\beta m/2}}{e^{\beta m} - 1}\frac{\cos(mt_{12})}{m}.$$
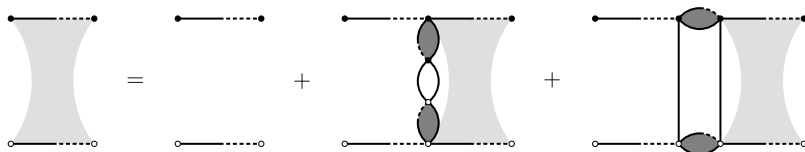
No chaos in the symmetric model

- The leading corrections to the averaged OTOC in the $O(N)$-symmetric model are described by the so-called **"ladder" diagrams**

- Substituting the **exponential ansatz** $C_{12;34} \sim e^{2\kappa t}$, $t = \frac{1}{2}(t_1 + t_2 - t_3 - t_4)$, into the Bethe-Salpeter equation, we get the equation on κ:

$$1 \approx \frac{64}{N} \frac{w^2 \lambda^2}{\tilde{m}^6} \frac{1}{\mu^4} \frac{1}{\left(1 + \frac{\kappa^2}{\tilde{m}^2}\right)^2} + \frac{4}{N} \frac{w^2 \lambda^2}{\tilde{m}^6} \frac{5 + \frac{\kappa^2}{\tilde{m}^2}}{\left((\mu+1)^2 + \frac{\kappa^2}{\tilde{m}^2}\right)\left((\mu-1)^2 + \frac{\kappa^2}{\tilde{m}^2}\right)\left(1 + \frac{\kappa^2}{\tilde{m}^2}\right)},$$

where $w = e^{\beta \tilde{m}/2}/\left(e^{\beta \tilde{m}} - 1\right)$, $\tilde{m}$ is the renormalized mass, and μ is the parameter of the resummed vertices (shaded bubbles on the picture)

- All solutions to this equation are purely inaginary; hence, there is **no quantum chaos in the $O(N)$-symmetric model**

Chaos in the full nonsymmetric model

- Keeping in mind the leading contributions from nonsymmetric vertices and using the same ansatz for $\tilde{C}_{12;34}$, we get the equation on κ in the full model:

$$\frac{1}{\alpha^2} \approx -\frac{1536}{N^2} \frac{w^2\lambda^2}{\tilde{m}^6} \frac{1}{\mu^6} \frac{1}{\left(1 + \frac{\kappa^2}{\tilde{m}^2}\right)^2} - \frac{24}{N^2} \frac{w^2\lambda^2}{\tilde{m}^6} \frac{\left(5 + \frac{\kappa^2}{\tilde{m}^2}\right)\left(3\mu^2 - 3 + (\mu^2 + 6)\kappa^2 + \kappa^4\right)}{\left(1 + \frac{\kappa^2}{\tilde{m}^2}\right)\left((\mu+1)^2 + \frac{\kappa^2}{\tilde{m}^2}\right)^2 \left((\mu-1)^2 + \frac{\kappa^2}{\tilde{m}^2}\right)^2}$$

- The solutions to this equation **has a positive real part**

- The maximal quantum Lyapunov exponent is as follows:

$$\kappa_q = \alpha \frac{8\sqrt{6}}{\mu^3} \frac{w\lambda}{\tilde{m}^3} \frac{\tilde{m}}{N}$$

- The exponent scales as $\kappa_q \sim \sqrt[4]{\lambda/\beta}$ in the high-temperature limit and is exponentially suppressed in the low-temperature limit:

$$\kappa_q^{\text{high}} \approx \frac{4}{3}\frac{\alpha}{N}\sqrt[4]{\frac{\lambda}{\beta}}, \qquad \kappa_q^{\text{low}} \approx \sqrt{6}\frac{\alpha}{N}\frac{\lambda}{m^3}m \exp\left(-\frac{\beta m}{2}\right)$$

Comparison to classical chaos

- The $O(N)$-**symmetric** model is clearly **classically integrable**, i.e., its maximal Lyapunov exponent is zero

- The numerical calculations of the maximal Lyapunov exponent in the **nonsymmetric** model yield the following **high-temperature behavior**:

$$\bar{\kappa}_{cl} \approx (1.3 \pm 0.2) \, \frac{1}{N^{1.18 \pm 0.05}} \left(\frac{\lambda}{\beta}\right)^{0.28 \pm 0.02},$$

where we assume $\alpha = 1$ and use the relation $\beta \sim N/E$

- Details of this calculations are discussed in **Nikita Kolganov's poster**, which I kindly ask you to examine

- Thus, in both models, classical and quantum Lyapunov exponents **approximately coincide with each other**

Qualitative analysis

- In fact, the high-temperature behavior of classical and quantum Lyapunov exponents can be **deduced from dimensional grounds**

- In the limit $\beta m \ll 1$ and $\beta m \ll \lambda/m^3$, the quadratic part of the potential energy is negligible, so the Hamiltonian acquires the following form ($\alpha = 1$):

$$H^{\text{high}} \approx \sum_{i=1}^{N} \frac{1}{2}\pi_i^2 + \frac{\lambda}{4N} \sum_{i \neq j} \phi_i^2 \phi_j^2$$

- This "pruned" Hamiltonian is **invariant under the scale transformations**:

$$t \to \gamma^{-1} t, \qquad \phi_i \to \gamma \phi_i, \qquad H \to \gamma^4 H$$

- Since the Lyapunov exponent has the dimension of inverse time, this invariance implies the high-temperature dependence $\kappa \sim \sqrt[4]{E} \sim \sqrt[4]{\lambda/\beta}$

Analogy to billiards

- Furthermore, we can compare the **constant potential energy surface (CPE surface)** with a wall of a **Sinai billiard**

- It is known that Sinai billiards exhibit a chaotic behavior in the presence of **concave walls**

- In the nonsymmetric model, the CPE surface becomes concave at energies $E > E_{\mathsf{con}} = 3Nm^4/2\lambda$, which **agrees with the emergence of chaos**
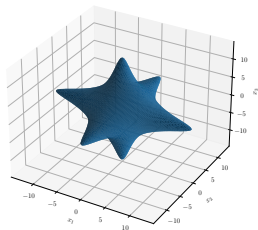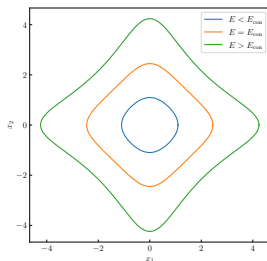

Figure: [Left] CPE curve for $N = 2$ and $E < E_{\mathsf{con}}$ (blue line), $E = E_{\mathsf{con}}$ (orange line), $E > E_{\mathsf{con}}$ (green line). [Right] CPE surface for $N = 3$ and $E \gg E_{\mathsf{con}}$.

Conclusion

- We suggest a tractable chaotic model — the nonlinear vector mechanics with a quartic interaction and thermal initial state
- In the $O(N)$-symmetric case, both classical and quantum Lyapunov exponents are zero
- In the nonsymmetric case, both exponents emerge in the high-temperature limit, approximately coincide, and scale as $\kappa_q \sim \kappa_{cl} \sim \frac{1}{N} \sqrt[4]{\frac{\lambda}{\beta}}$
- This calculation supports the use of OTOCs as a diagnostic of quantum chaos