Relativistic $GL(NM,C)$ Gaudin models on elliptic curve

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Abstract

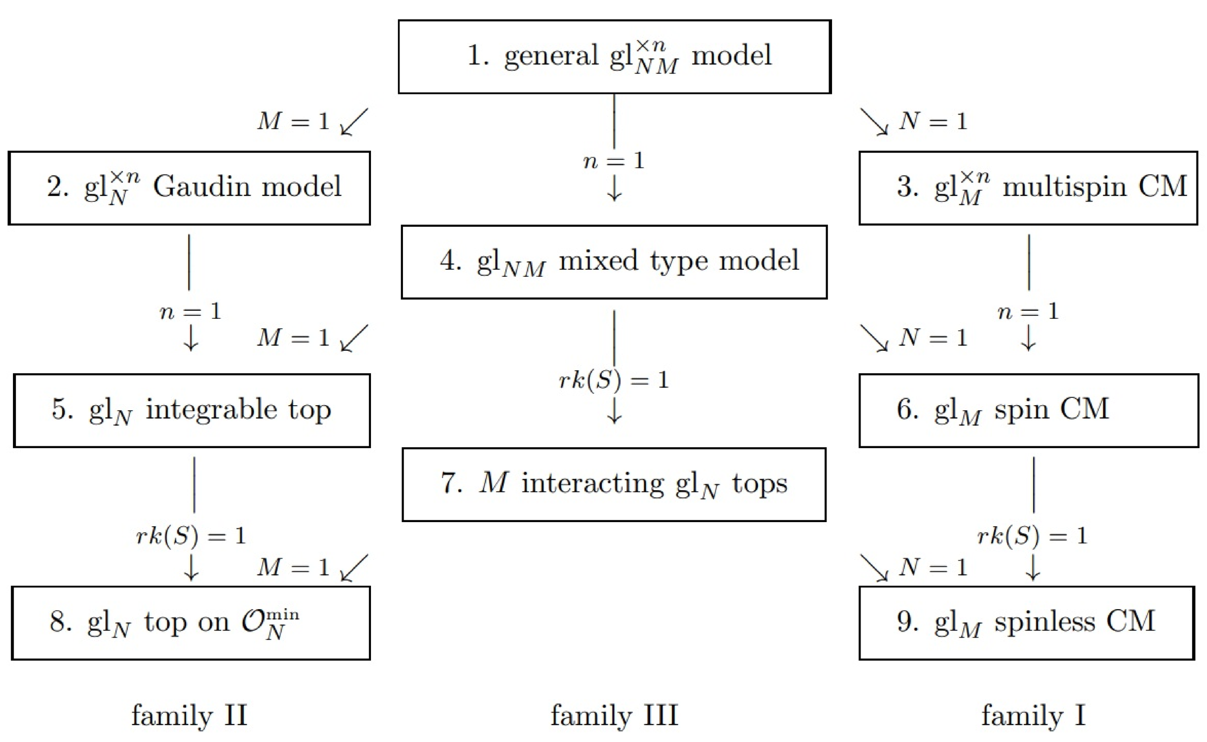
We present a classification for relativistic Gaudin models on GL-bundles over elliptic curve. We describe the most general $GL(NM)$ classical elliptic finite-dimensional integrable system, in which Lax matrix has n simple poles on elliptic curve. Also, we provide a description of this model through R-matrices satisfying associative Yang-Baxter equation. Finally, we describe the inhomogeneous Ruijsenaars chain and show that it can be considered as a particular case of multispin Ruijsenaars-Schneider model. This poster is based on the paper [1].

1. Non-relativistic systems

In our previous paper [2], we reviewed the non-relativistic classical integrable systems, which are governed by linear classical r-matrix structures on elliptic curve. Let us briefly outline their main features:

- The phase space consists of the many-body component $\mathbb{C}^{2M}$ and also of n copies of spin space parameterized by variables S^a_{ij}, $a = \overline{1,n}$, $i,j = \overline{1,N}$ treated as **the classical spin variables**. They are naturally arranged into $gl(N,\mathbb{C})$ valued matrix S^a.
- The spin component of the phase space is a reduced coadjoint orbit of $GL(N,\mathbb{C})$ Lie group: $O//H$. Instead of dealing with the reduced system, one can describe system on **unreduced phase space**. Such a system is non-integrable (it has additional terms in Lax equation) unless we impose some **additional constrains**.
- The classification scheme for the considered models is as follows:

Classification scheme for (spin) Calogero and Gaudin type models:

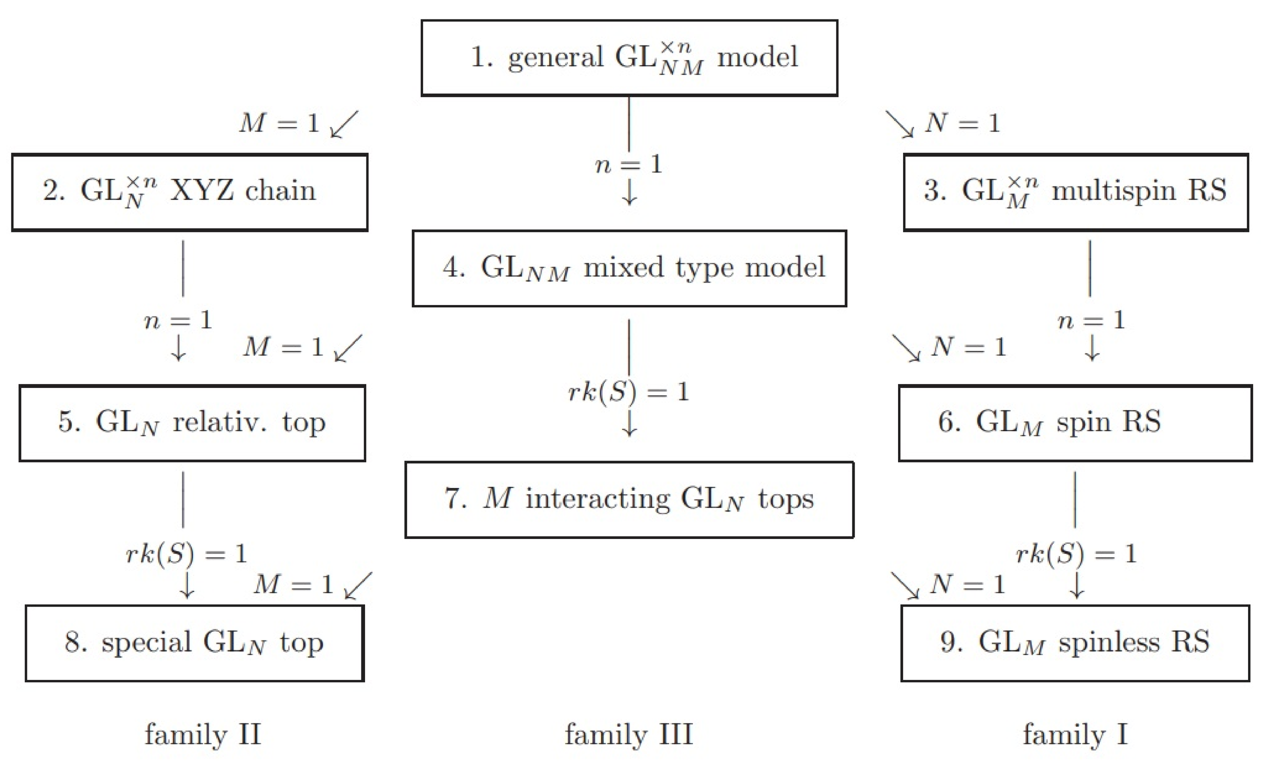


The $gl^{\times n}_{NM}$ system in the box 1 is **the most general model** in this class of systems. All other systems can be considered as particular cases of this one.

2. Relativistic generalizations

The described models admit relativistic extensions. In paper [1], we discuss the **relativistic analogue** of the above scheme. The classification is presented on the following scheme:

Classification scheme for elliptic relativistic models:



The top of this scheme is the model described by GL_{NM}-valued matrices with n poles on the elliptic curve (**box 1**). Describing it, we face a problem: the Poisson and r-matrix structures for the **spin elliptic Ruijsenaars-Schneider (RS) model** (RS model is widely known as a relativistic extension of Calogero-Moser system) are still unknown. This is why we deal with **the Lax equation** only and do not provide a Hamiltonian description.

The Lax pair for the general model has a block-matrix structure with $N \times N$ blocks:

$$
\begin{gathered}
\mathcal{L}(z) = \sum_{i,j=1}^{M} E_{ij} \otimes \mathcal{L}^{ij}(z) \in \mathrm{Mat}(NM,\mathbb{C}), \quad \mathcal{L}^{ij}(z) \in \mathrm{Mat}(N,\mathbb{C}),\\
\mathcal{L}^{ij}(z) = \sum_{\alpha}\sum_{a=1}^{n} T_\alpha \mathcal{S}^{ij,a}_\alpha \varphi_\alpha\Big(z - z_a, \omega_\alpha + \frac{q_{ij}+\eta}{N}\Big),\\
\mathcal{M}^{ij}(z) = -\delta_{ij}\sum_{a=1}^{n} T_0 \mathcal{S}^{ii,a}_{0,0}\Big(E_1(z-z_a) + E_1\Big(\frac{\eta}{N}\Big)\Big) - \delta_{ij}\sum_{\alpha\neq 0}\sum_{a=1}^{n} T_\alpha \mathcal{S}^{ii,a}_\alpha \varphi_\alpha(z-z_a,\omega_\alpha) -\\
-(1-\delta_{ij})\sum_{\alpha}\sum_{a=1}^{n} T_\alpha \mathcal{S}^{ij,a}_\alpha \varphi_\alpha\Big(z - z_a, \omega_\alpha + \frac{q_{ij}}{N}\Big).
\end{gathered}
\tag{1}
$$

Here $q_{ij} = q_i - q_j$; T_α **is a special matrix basis** in $\mathrm{Mat}(NM,\mathbb{C})$, $\alpha = (\alpha_1,\alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$; $E_1(z)$ is the **Eisenstein function** and φ is the modified **Kronecker elliptic function** $\varphi_\alpha(z,\omega_\alpha+u) = \exp(2\pi i\frac{\alpha_2}{N}z)\phi(z,\omega_\alpha+u)$. Our main result is the following

Theorem 1. *The Lax equation with additional term*

$$
\frac{d}{dt}\mathcal{L}(z) = [\mathcal{L}(z),\mathcal{M}(z)] + \sum_{i,j=1}^{M}\sum_{c=1}^{n}\sum_{\alpha}(\mu_i-\mu_j)\mathcal{S}^{ij,c}_\alpha f_\alpha\Big(z - z_c,\ \omega_\alpha + \frac{q_{ij}+\eta}{N}\Big) E_{ij}\otimes T_\alpha
$$

where

$$
\mu_i = \frac{\dot q_i}{N} - \sum_{a=1}^{n}\mathcal{S}^{ii,a}_{0,0}, \quad i = 1,\dots,M
$$

for the Lax pair (1) is equal to the equations of motion:

$$
\begin{gathered}
\dot{\mathcal{S}}^{ij,a} = \mathcal{S}^{ij,a}J^{\eta}(\mathcal{S}^{jj,a}) - J^{\eta}(\mathcal{S}^{ii,a})\mathcal{S}^{ij,a} + \sum_{k\neq j}\mathcal{S}^{ik,a}J^{\eta,q_{kj}}(\mathcal{S}^{kj,a}) - \sum_{k\neq i}J^{\eta,q_{ik}}(\mathcal{S}^{ik,a})\mathcal{S}^{kj,a} +\\
+\sum_{b:b\neq a}^{n}\mathcal{S}^{ij,a}\Big(\mathcal{S}^{ii,b}_{0,0} - \mathcal{S}^{jj,b}_{0,0}\Big)\Big(E_1\Big(\frac{\eta}{N}\Big) + E_1(z_{ab}) - \phi\Big(z_{ab},\frac{\eta}{N}\Big)\Big) +\\
+\sum_{b:b\neq a}^{n}\Big(\mathcal{S}^{ij,a}\tilde J^{\eta}_a(\mathcal{S}^{jj,b}) - \tilde J^{\eta}_a(\mathcal{S}^{ii,b})\mathcal{S}^{ij,a}\Big) + \sum_{b:b\neq a}^{n}\Big(\sum_{k\neq j}\mathcal{S}^{ik,a}\tilde J^{\eta,q_{kj}}_a(\mathcal{S}^{kj,b}) - \sum_{k\neq i}\tilde J^{\eta,q_{ik}}_a(\mathcal{S}^{ik,b})\mathcal{S}^{kj,a}\Big),
\end{gathered}
$$

where J, $\tilde J$ are generalized inverse inertia tensors. The Lax equation holds true on the constraints: $\mu_i = 0$.

3. R-matrix description

All models from our classification can be described in terms of **R-matrices**, satisfying the **associative Yang-Baxter equation**:

$$
R^z_{12}R^w_{23} = R^w_{13}R^{z-w}_{12} + R^{w-z}_{23}R^z_{13}, \quad R_{ab} = R_{ab}(q_a - q_b).
$$

We also require some additional properties — unitary condition and skew-symmetry.

The R-matrix formulation is pretty convenient. Different R-matrices correspond to different rational and trigonometric limits of the classification scheme. The elliptic version is given by **the Belavin-Baxter elliptic R-matrix**.

For the $\mathrm{GL}^{\times n}_{NM}$ model, we have the following R-matrix Lax pair

$$
\begin{gathered}
\mathcal{L}^{ij}(z) = \sum_{a=1}^{n}\mathrm{tr}_2(R^{z-z_a}_{12}(q_{ij}+\eta)P_{12}S^{ij,a}_2),\\
\mathcal{M}^{ij}(z) = -\delta_{ij}\sum_{a=1}^{n}\mathrm{tr}_2(R^{z-z_a,(0)}_{12}P_{12}S^{ii,a}_2) - (1-\delta_{ij})\sum_{a=1}^{n}\mathrm{tr}_2(R^{z-z_a}_{12}(q_{ij})P_{12}S^{ij,a}_2)\,.
\end{gathered}
\tag{2}
$$

Theorem 2. *The Lax equation with additional term:*

$$
\frac{d}{dt}\mathcal{L}(z) = [\mathcal{L}(z),\mathcal{M}(z)] + \sum_{i,j=1}^{M}\sum_{a=1}^{n}\mathrm{tr}_2\Big((\mu_i-\mu_j)F^{z-z_a}_{12}(q_{ij}+\eta)P_{12}\,\mathcal{S}^{ij,a}_2\Big)
$$

where $F^z_{12}(q) = \partial_q R^z_{12}(q)$ and

$$
\mu_i = \dot q_i - N\sum_{a=1}^{n}\mathcal{S}^{ii,a}_{0,0} = \dot q_i - \sum_{a=1}^{n}\mathrm{tr}(\mathcal{S}^{ii,a})\,, \quad i = 1,\dots,M
$$

for the Lax pair (2) is equivalent to the equations of motion:

$$
\begin{gathered}
\dot{\mathcal{S}}^{ij,a} = \mathcal{S}^{ij,a}\mathcal{J}^{\eta}(\mathcal{S}^{jj,a}) - \mathcal{J}^{\eta}(\mathcal{S}^{ii,a})\mathcal{S}^{ij,a} + \sum_{k\neq j}\mathcal{S}^{ik,a}\mathcal{J}^{\eta,q_{kj}}(\mathcal{S}^{kj,a}) - \sum_{k\neq i}\mathcal{J}^{\eta,q_{ik}}(\mathcal{S}^{ik,a})\mathcal{S}^{kj,a} +\\
+\sum_{b:b\neq a}^{n}(\mathcal{S}^{ij,a}\tilde{\mathcal{J}}^{\eta}_a(\mathcal{S}^{jj,b}) - \tilde{\mathcal{J}}^{\eta}_a(\mathcal{S}^{ii,b})\mathcal{S}^{ij,a}) + \sum_{b:b\neq a}^{n}\Big(\sum_{k\neq j}\mathcal{S}^{ik,a}\tilde{\mathcal{J}}^{\eta,q_{kj}}_a(\mathcal{S}^{kj,b}) - \sum_{k\neq i}\tilde{\mathcal{J}}^{\eta,q_{ik}}_a(\mathcal{S}^{ik,b})\mathcal{S}^{kj,a})\Big)
\end{gathered}
$$

Here, generalized inverse inertia tensors J, $\tilde J$ are expressed in terms of R-matrices. The Lax equation holds true on the constraints: $\mu_i = 0$.

4. Inhomogeneous Ruijsenaars chain

The top model in case of minimal rank is gauge equivalent to **the spinless Calogero-Moser model**. The same thing happens with their relativistic deformations — the relativistic top of minimal rank and RS model:

$$
L^{rel\ top}(S,z) = g(z,q)L^{\mathrm{RS}}(z)g^{-1}(z,q).
$$

Gauge equivalence allows us to obtain a parametrization of spin variables by canonical ones.

We considered an **inhomogeneous version of RS chain**:

$$
\begin{gathered}
T(z) = L^{N\eta}(S^1, z - z_1)L^{N\eta}(S^2, z - z_2)\dots L^{N\eta}(S^n, z - z_n),\\
\tilde L^k_{ij}(z) = \phi(z, \bar q^{k-1}_i - \bar q^k_j + \eta)\frac{\prod_{l=1}^{N}\vartheta(\bar q^k_j - \bar q^{k-1}_l - \eta)}{\vartheta(-\eta)\prod_{l:l\neq j}^{N}\vartheta(\bar q^k_j - \bar q^k_l)}e^{p^k_j/c}.
\end{gathered}
$$

We performed similar **gauge transformation** of monodromy matrix: $\tilde T(z) = G^{-1}T(z)G$. It allows us to calculate explicit form of non-local Hamiltonians: $H_i = \operatorname*{Res}_{z=z_i}\mathrm{tr}\,\tilde T(z)$. Representing $\tilde T(z)$ in the additive form, we obtain the multispin RS model:

$$
\tilde T_{ij}(z) = \sum_{k=1}^{n}\mathcal{S}^k_{ij}\phi(z - z_k, q^1_i - q^1_j + n\eta).
$$

From the explicit form of $\tilde T$, we see that spin variables are rank one matrices. Positions of particles (q^1_i) and spin variables are parameterized by the rest of canonical variables.

References

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In memory of Bonya, the most intrepid pug