Rotating black holes

Kerr metric

$$ds^2 = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 - \frac{4mra\sin^2\theta}{\rho^2} dt\, d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

$$\begin{aligned} \Delta(r) &= r^2 - 2mr + a^2 \\ \rho(r,\theta)^2 &= r^2 + a^2\cos^2\theta \\ \Sigma(r,\theta) &= (r^2+a^2)^2 - \Delta(r) a^2 \sin^2\theta \end{aligned}$$

- Two free parameters m and a (rotation parameter), $J = Ma$.
- This metric is time-independent and axially symmetric, with the two commuting Killing vectors $\xi = \partial_t$, $\eta = \partial_\phi$.
- Stationary (but not static) $\Rightarrow$ there is a term $g_{t\phi}$, no symmetry $t \to -t$ (More invariantly, this is the statement that the Killing vector t is not hypersurface-orthogonal)
- It reduces to the Schwarzschild metric for $a = 0$. Also reduces to the Minkowski spacetime when $m = 0$, but in 'weird' rotating coordinates.
- Electric charge can be added to this solution by the same replacement $m \to m - Q^2/(2r)$.

Kerr metric

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$$\Delta(r) = r^2 - 2mr + a^2$$
$$\rho(r,\theta)^2 = r^2 + a^2\cos^2\theta$$
$$\Sigma(r,\theta) = (r^2 + a^2)^2 - \Delta(r) a^2 \sin^2\theta$$

- A symmetry $\theta \to \theta - \pi$. Also a symmetry $(t, \phi) \to (-t, -\phi)$, related to *circularity* property of the solution. (Running backwards in time negative spin = running forward in time with positive spin.)

$$\xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(t)} = \xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(\varphi)} = 0\,.$$

- Asymptotically

$$ds^2 \simeq -\left(1 - \frac{2m}{r}\right) dt^2 - \frac{4ma\sin^2\theta}{r} dt d\phi + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Kerr black hole: ergosphere

$$ds^2 = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 - \frac{4mra\sin^2\theta}{\rho^2} dt\, d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

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- Static observers, remaining at fixed values of the spatial coordinates (r, θ, ϕ), with 4-velocity

$$u^\alpha \sim \xi^\alpha$$

- A static limit or infinite redshift surface for such observers?

$$\xi^2 = g_{tt} = 0$$

$$g_{tt}(r,\theta) = 0 \quad \Leftrightarrow \quad \rho^2 - 2mr = r^2 + a^2\cos^2\theta - 2mr = 0$$

$$r_{sl}(\theta) = m + \sqrt{m^2 - a^2\cos^2\theta}$$

Kerr black hole: ergosphere

- No curvature singularity at $r = r_{sl}$.
- It defines the static limit surface for static observers: no static observers can exist for $r < r_{sl}$
- Also defines a surface of infinite redshift for static observers.
- Killing vector $\xi = \partial_t$ becomes null.
- For the Schwarzschild metric, $r_{sl} \to 2m = r_g$ reduces to the Schwarzschild radius.

However this surface is not a horizon

Kerr black hole: ergosphere

- Although no static observers can exist for $r < r_{sl}$ this does not by itself imply that one cannot escape from that region. Stationary observers can escape with $u^\alpha \sim \xi^\alpha + \Omega \eta^\alpha$.

- This surface is timelike, it has a spacelike normal.

$$S(r,\theta) = r - r_{sl}(\theta) = 0$$

Normal vector $\qquad N_\alpha = \partial_\alpha S : \quad N_\alpha = (0, 1, -dr_{sl}/d\theta, 0)$

$$N_\alpha N^\alpha = g^{rr} + g^{\theta\theta}(dr_{sl}(\theta)/d\theta)^2 = \frac{1}{2mr_{sl}} \frac{m^2 a^2 \sin^2\theta}{m^2 - a^2\cos^2\theta} \geq 0$$

Clearly cannot be a horizon

Kerr black hole: inside the ergosphere

- Stationary observers $u^\alpha \sim \xi^\alpha + \Omega \eta^\alpha$.
- u^α is timelike when $g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} < 0$

$$\Omega_-(r,\theta) < \Omega(r,\theta) < \Omega_+(r,\theta)$$

$$\Omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}}{g_{\phi\phi}}$$

$$\Omega_- \quad \begin{cases} < 0 & \text{for} \quad r > r_{sl} \\ = 0 & \text{for} \quad r = r_{sl} \\ > 0 & \text{for} \quad r < r_{sl} \end{cases}$$

- Outside the ergosphere stationary observers can rotate either with or against the rotation of the black hole. On and inside the ergosphere a stationary observer only rotate with (i.e. to be dragged along by) the black hole.
- One can add a small motion in the direction of r or θ and see that one can escape from the inside of the ergosphere, as long as stationary observers exist.

Kerr black hole: Horizon

- Stationary observers cease to exist when

$$g_{t\phi}^2 - g_{tt}g_{\phi\phi} = \Delta(r)\sin^2\theta \ = 0 \qquad \text{also} \quad g^{rr} = \frac{\Delta}{\rho^2}$$

$$\Delta(r) = r^2 - 2mr + a^2 = 0 \qquad \Rightarrow \qquad r_\pm = m \pm \sqrt{m^2 - a^2}$$

- We note that $\quad r_+ = m + \sqrt{m^2 - a^2} \leq m + \sqrt{m^2 - a^2\cos^2\theta} = r_{sl}(\theta)$

  Equality is only at the poles $\theta = 0, \pi$

- Hypersurface of constant r with the normal $N_\alpha \sim \partial_\alpha r$ becomes null at r_+:

$$g^{\alpha\beta}\partial_\alpha r \partial_\beta r = 0$$

Kerr black hole: Horizon

- At $r = r_+$ the stationary observers have the angular velocity

$$\Omega_h \equiv \omega(r_+) = \frac{a}{r_+^2 + a^2}$$

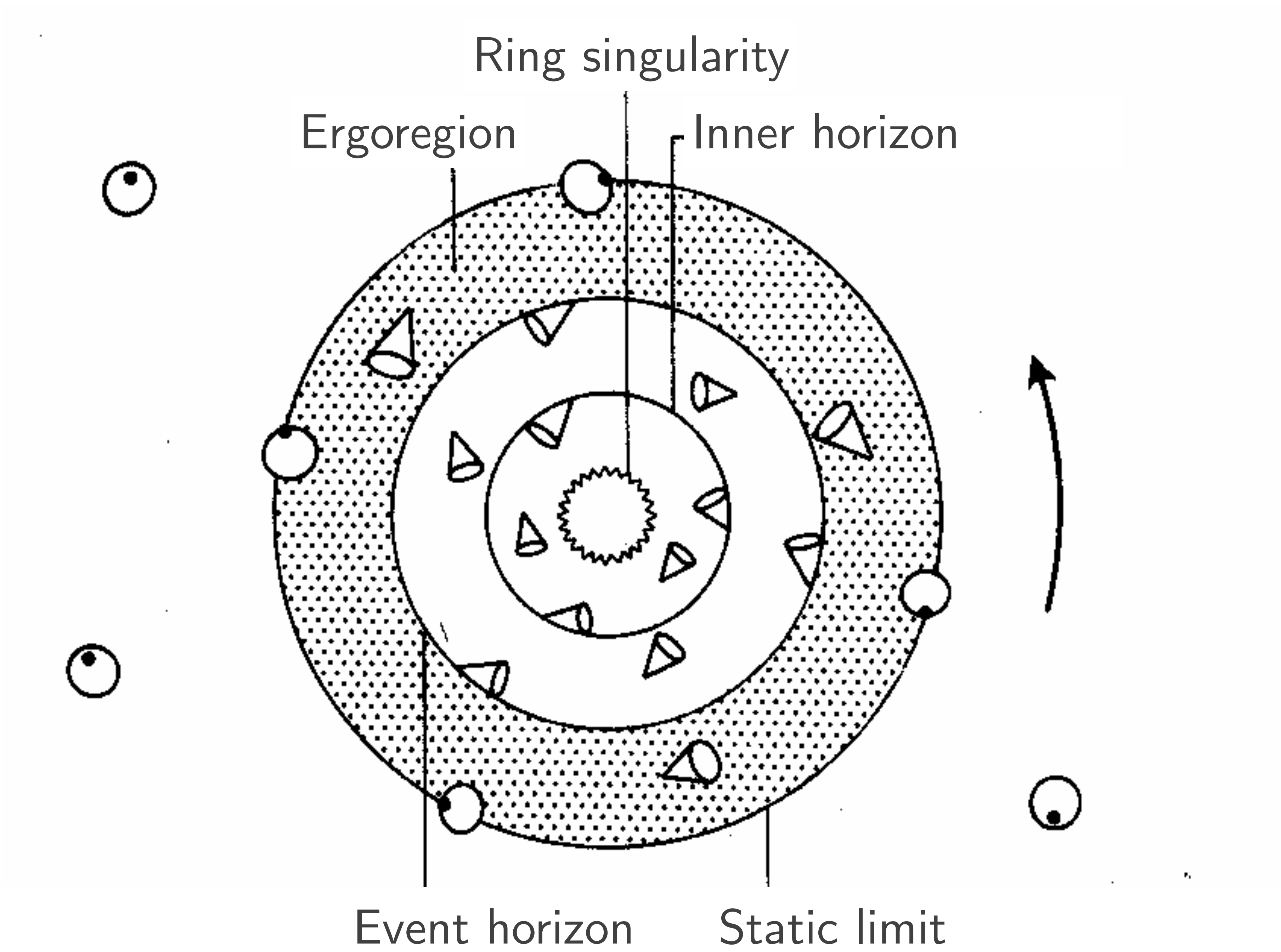
- The Killing vector corresponding to the stationary observers there is null

$$\xi_h \equiv \xi_{\Omega_h} = \xi + \Omega_h \eta \qquad\qquad \Rightarrow \qquad\qquad \left(g_{\alpha\beta}\xi_h^\alpha \xi_h^\beta\right)|_{r=r_+} = 0$$

- This surface is a Killing horizon and also the event horizon

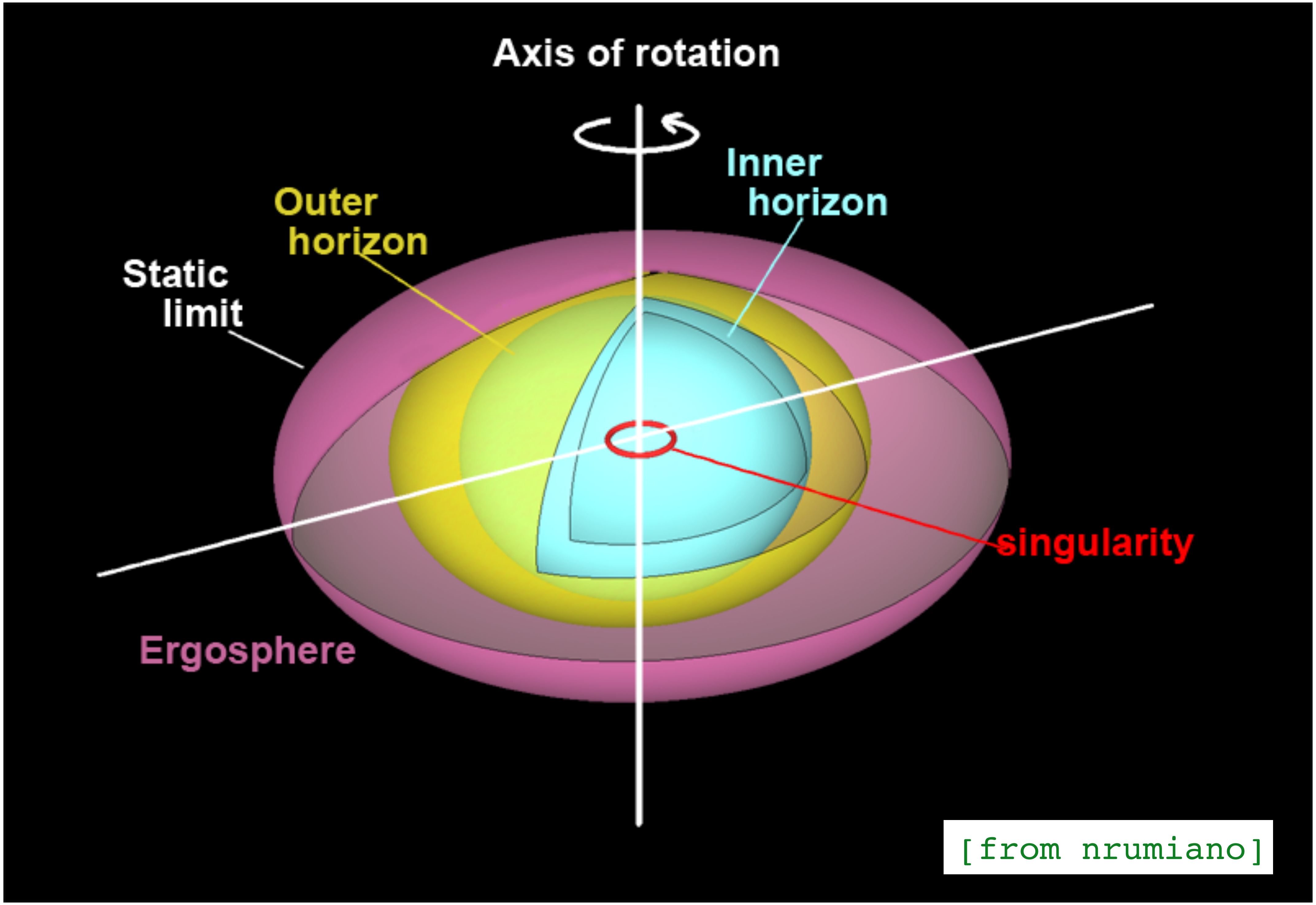
- Singularity occurs at $\rho^2 = r^2 + a^2\cos^2\theta = 0 \Rightarrow r = \cos\theta = 0$ It is a ring singularity, $x^2 + y^2 = a^2$, $z = 0$.

Important surfaces in the Kerr metric



[from d'Inverno's book]

Kerr summary



Killing horizon vs event horizon

- Rigidity theoreom (`Hawking`): The event horizon $\mathcal{H}$ of a real analytic, stationary, regular, vacuum spacetime is a Killing horizon: $\exists$ a Killing field k normal to $\mathcal{H}$ which verifies $k^2 = 0$ on $\mathcal{H}$.

- For the outer horizon of the Kerr spacetime, this Killing vector is

$$k = \partial_t + \frac{a}{2Mr_+}\partial_\varphi$$

- One can define the surface gravity κ_+ of $\mathcal{H}$ as

$$k^\mu \nabla_\mu k^\nu = \kappa_+ k^\nu$$

- The surface gravity is constant on $\mathcal{H}$ and is related to the Hawking temperature $T_H = \kappa_+/2\pi$

Killing horizon vs event horizon

Stationary,
electro-vacuum,
analytic,
non-degenerate,
connected,
regular black hole
=
Kerr-Newman
in the exterior region

talk by Piotr T. Chruściel

Kerr singularity, extremal black holes, time machines

- For $a > M$ there is no horizon $\Rightarrow$ naked singularity
- Extremal black hole $a = M$
- Time machine inside Kerr black hole

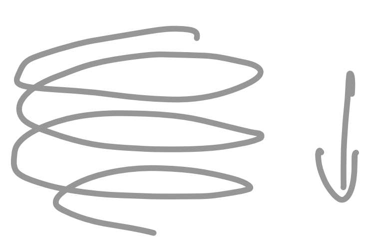
$$ds^2 = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 - \frac{4mra\sin^2\theta}{\rho^2} dt\, d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

$$\Sigma = \left(r^2 + a^2\right)^2 - \Delta(r) a^2 \sin^2\theta = \left(r^2 + a^2\right)\rho^2 + 2mra^2\sin^2\theta$$

Consider ρ close to 0, i.e. close to the ring singularity.

There is a region $(r < 0)$ where the sign changes.

$\partial/\partial\phi$ is timelike. Consider now $t = const$, $r = const$, $\theta = const$, and motion along $\phi-$direction $\Rightarrow$ closed timelike curves.

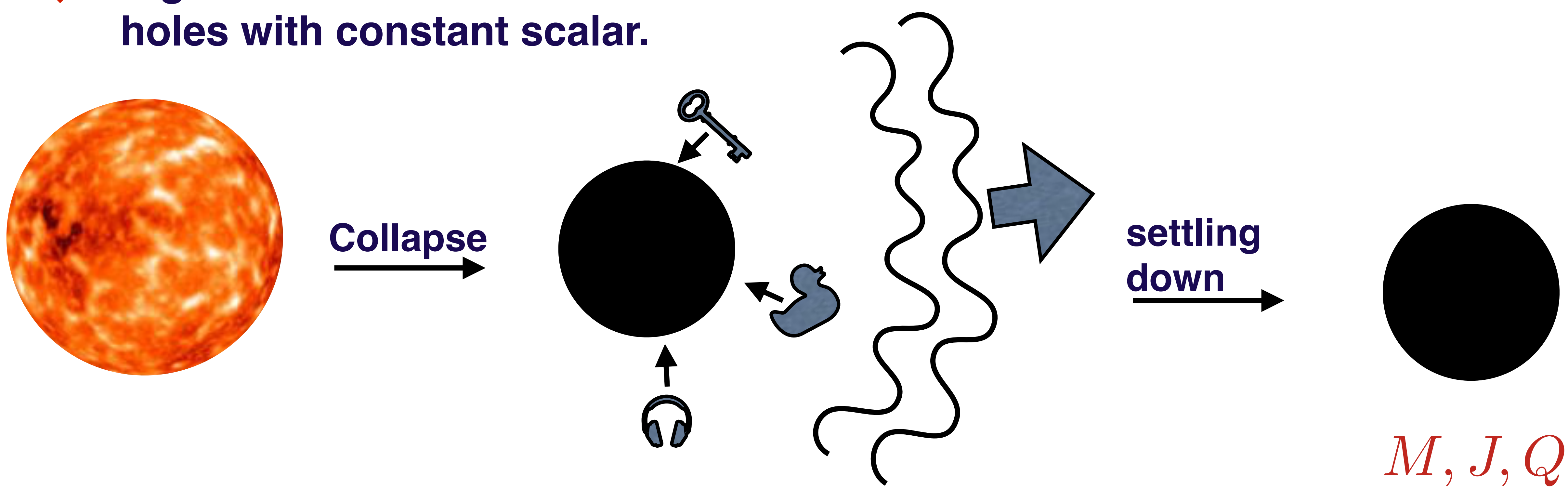


Decreasing $t \Rightarrow$ time machine

Black holes are bald

- Gravitational collapse...
- Black holes eat or expel surrounding matter
- Their stationary phase is characterised by a limited number of charges
- No details about collapse
- Black holes are bald

- **No hair theorems/arguments dictate that adding degrees of freedom lead to trivial (General Relativity) or singular solutions.**
- **E.g. in the standard scalar-tensor theories BH solutions are GR black holes with constant scalar.**



Example of hairy black hole

BBMB solution

Bocharova et al'70, Bekenstein'74

Conformally coupled scalar field:

$$S[g_{\mu\nu}, \phi] = \int \sqrt{-g} \left(\frac{R}{16\pi G} - \frac{1}{2}\partial_\alpha \phi \partial^\alpha \phi - \frac{1}{12} R\phi^2 \right) d^4x$$

Static spherically symmetric (nontrivial) solution:

$$ds^2 = -\left(1 - \frac{m}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{m}{r}\right)^2} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Secondary scalar hair:

$$\phi = \sqrt{\frac{3}{4\pi G}} \frac{m}{r - m}$$

NB. The geometry is of that of extremal RN.

The scalar field is unbounded at r=m

Gauss-Bonnet term

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \lambda\phi\hat{G}\right]$$

Gauss-Bonnet invariant: $\hat{G} = R_{\mu\nu\sigma\alpha}R^{\mu\nu\sigma\alpha} - 4R_{\mu\nu}R^{\mu\nu} + R^2$

Horndeski theory with $G_5 \propto \ln|X| \Rightarrow$ assumption (iii) is violated.

EoM for the scalar: $\Box\phi = -\lambda\hat{G}$

Source for the scalar: it cannot be trivial in BH background

Campbell et al'92
Kanti et al'96
Sotiriou and Zhou'13

Solutions in "Galileon theory"

EB, Charmousis'13

$$\mathcal{L}^{\Lambda\mathrm{CGJ}} = R - \eta(\partial\phi)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\Lambda.$$

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$$\boxed{\phi = qt + \psi(r)}$$

Time-dependent scalar !

Asymptotically dS/AdS:

$$f = h = 1 - \frac{\mu}{r} - \frac{\Lambda_{\text{eff}}}{3}r^2, \qquad \psi' = \pm\frac{q}{h}\sqrt{1-h}, \qquad \Lambda_{\text{eff}} = -\frac{1}{2\beta}$$

Properties of disformed Kerr in modified gravity

EB, T.Anson, C.Charmousis, M.Hassaine'20

- *Ergosphere (static limit)*: static timelike observers can no longer exist, the Killing vector $l^\mu = (1,0,0,0)$ becomes null. I.e. $\tilde{g}_{tt} = 0$, or

$$\tilde{g}_{tt} = 0 \quad \Rightarrow \boxed{r_E = \tilde{M} + \sqrt{\tilde{M}^2 - a^2\cos^2\theta}}$$

- *Stationary limit.* Observers constant (r,θ), with a 4-velocity $u = \partial_t + \omega\partial_\varphi$. These observers cease to exist at the surface $\tilde{g}_{tt}\tilde{g}_{\varphi\varphi} - \tilde{g}^2_{t\varphi} = 0$, i.e.

$$\boxed{P(r,\theta) \equiv r^2 + a^2 - 2\tilde{M}r + \frac{2\tilde{M}Da^2 r\sin^2\theta}{\rho^2(r,\theta)} = 0}$$

The surface is *timelike* and thus cannot be a horizon.

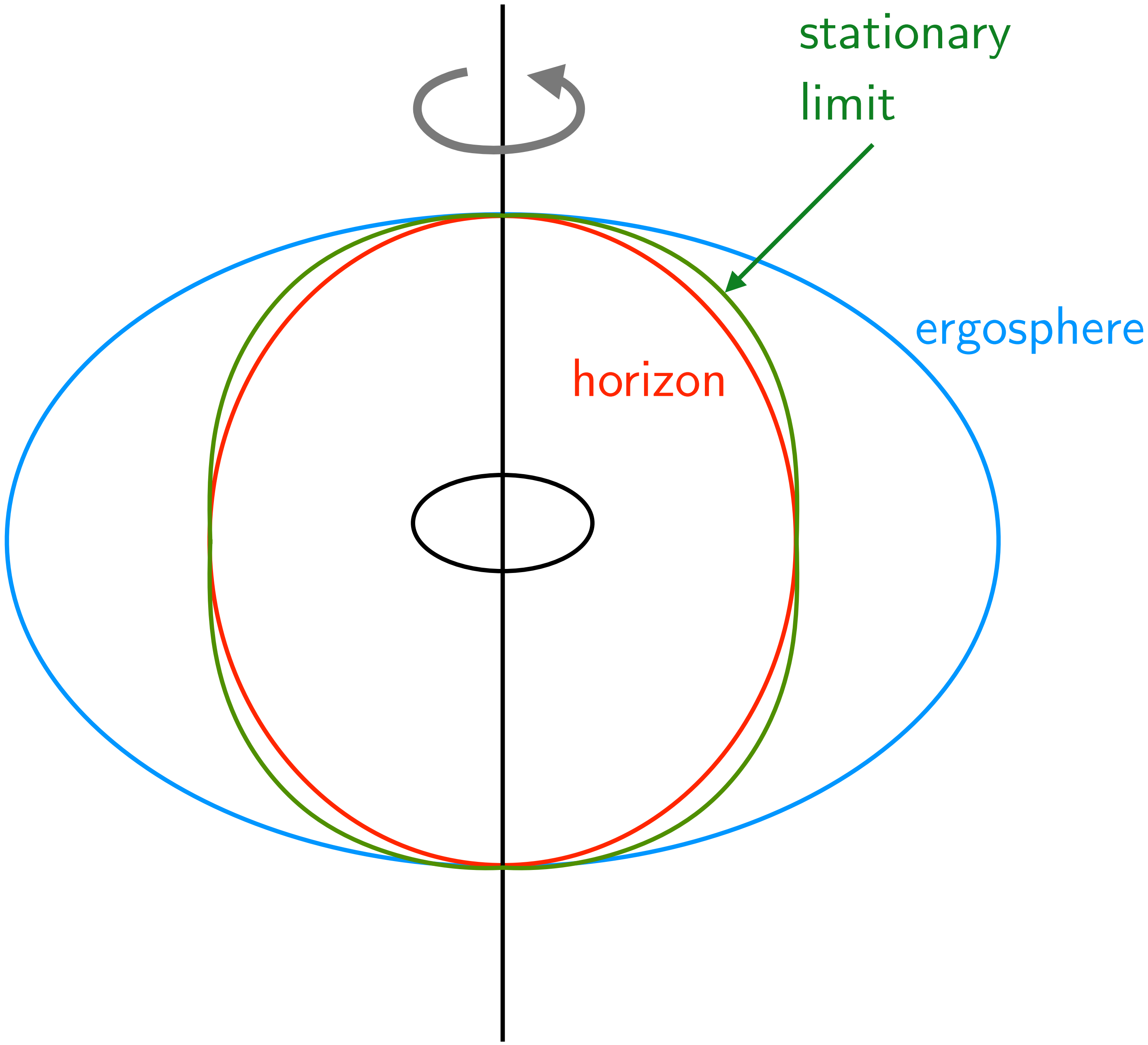
- *Horizon*: a null hypersurface of the form $r = R(\theta)$. The normal has components

$$n_\mu = (0, 1, -R'(\theta), 0)$$

The condition $n^2 = 0$ yields

$$\boxed{R'(\theta)^2 + P(R,\theta) = R'(\theta)^2 + R^2 + a^2 - 2\tilde{M}R + \frac{2\tilde{M}Da^2 R\sin^2\theta}{\rho^2(R,\theta)} = 0}$$

Surfaces



Gravitational waves

- The solutions considered so far do not reveal dynamical degrees of freedom of the gravity theory.
- Even collapsing shell solution does not involve dynamics of gravity. Dynamics of the solution is fully determined by the matter degrees of freedom, i.e. by the dynamics of the collapsing shell.
- Gravitational waves do contain dynamical degrees of freedom of GR.

Degrees of freedom in GR

- $g_{\mu\nu}$ contains 10 independent components $\frac{D(D+1)}{2} = 10$.
- There is gauge freedom, $x^\mu \rightarrow \tilde{x}^\mu(x)$, we can kill 4 components, i.e. 10-4=6.

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- However in GR there are only 2 dynamical d.o.fs, corresponding to 2 polarization. Why the mismatch?
- There are constraints in the theory that can be revealed by considering the Hamiltonian formalism.

However it is difficult

Example of Maxwell field

- Maxwell field in flat spacetime

$$L_{EM} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- A_μ has 4 components
- Gauge invariance $A_\mu \to A_\mu + \partial_\mu \alpha$
- 4-1 $=$ 3 (should have 2)

- $A_\mu = (A_0, A_i^T + \partial_i \alpha)$, $\partial_i A_i^T = 0$. We can set $\alpha = 0$ by a gauge choice.
- 0-component: $-\ddot{A}_0 + \triangle A_0 - \frac{\partial}{\partial t}(-\dot{A}_0 + \partial_i A_i^T) = 0$

A_0 is physical but non-dynamical

Solution with a source charge, $A_\mu = \left(\frac{q}{4\pi r}, 0, 0, 0\right)$

Linearised gravity

$$g_{\mu\nu} = g^{(1)}_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} \qquad\qquad |h_{\mu\nu}| \ll 1$$

We will drop terms which are quadratic or of higher power in $h_{\mu\nu}$

$$g^{(1)\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$

the linearised vacuum Einstein equations $\quad G^{(1)}_{\mu\nu} = 0$

$$\Updownarrow$$

$$\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h = 0$$

Can be obtained from the quadratic Minkowski space field theory action for a free massless spin-2 field described by $h_{\mu\nu}$.

$$S[h_{\mu\nu}] = \int d^4x\, \mathcal{L}(h_{\mu\nu}, \partial_\sigma h_{\mu\nu})$$

$$\mathcal{L}(h_{\mu\nu}, \partial_\sigma h_{\mu\nu}) = -\tfrac{1}{4} h_{\mu\nu,\sigma} h^{\mu\nu,\sigma} + \tfrac{1}{2} h_{\mu\nu,\sigma} h^{\sigma\mu,\nu} + \tfrac{1}{4} h^{,\sigma} h_{,\sigma} - \tfrac{1}{2} h_{,\sigma} h^{\mu\sigma}{}_{,\mu}$$

Indeed $\qquad \delta S[h_{\mu\nu}] = -\int d^4x\, G^{(1)}_{\mu\nu}\delta h^{\mu\nu}$

Linearised Einstein Equations

In the presence of matter we have $$G^{(1)}_{\mu\nu} = 8\pi G_N T^{(0)}_{\mu\nu}$$

Only the zero'th order term in the h-expansion appears on the right hand side of this equation.

$$\partial_\mu T^{(0)\mu\nu} = 0$$

Conservation law

Compatible

$$\partial_\mu G^{(1)\mu\nu} = 0$$

Linearized Bianchi identity

Analog of the full Bianchi identity

$$\nabla_\mu G^{\mu\nu} = 0$$

Gauge freedom in linearised gravity

Diffeomorphism invariance of GR: we can freely choose coordinates without changing the form of equations of motion

$$x \to y = y(x) \qquad \bar{g}_{\alpha\beta}(y) = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta} g_{\mu\nu}(x)$$

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu$$

(Similar to gauge invariance of the Maxwell field $A_\mu \to A_\mu + \partial_\mu \alpha$)

The old and new $h_{\mu\nu}$ is taken at the same coordinates

Note relation to the Killing tensor

Classes of perturbations

$g_{\mu\nu} = \eta_{\mu\nu} \qquad \delta g_{\mu\nu} = h_{\mu\nu}$

Three types of perturbations: scalar, vector, tensor

h_{00} behaves as a scalar by such transformations

$h_{00} = -2\Phi$ (lapse N в ADM)

h_{0i} behaves as a vector (with respect to index i)

$h_{0i} = B_{,i} + S_i \;, \quad S^i_{\;,i} = 0$ (c.f. $A_\mu = A^T_\mu + \partial_\mu \alpha$)
(shift N_i)

$$h_{ij} = -2\psi\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h^{TT}_{ij}$$

$$F_{i,i} = 0 \;, \quad h^{TT} = 0 \;, \quad \partial_i h^{TT}_{ij} = 0$$

Classes of perturbations

We can choose gauge $E = B = 0$, $F_i = 0$

$10 - 4 = 6$

Linearized Einstein equations:

$$G_{00} = 2\Delta\Psi \quad (\Psi - \text{non-dynamical})$$

$$G_{0i} = 2\dot{\Psi}_{,i} - \frac{1}{2}\Delta S_i \quad (S - \text{nondynamical})$$

$$G_{ij} = (\Psi - \Phi)_{,ij} + \delta_{ij}\left[2\ddot{\Psi} + \Delta(\Phi - \Psi)\right] + \underbrace{\frac{1}{2}\left[\ddot{h}^{TT}_{ij} - \Delta h^{TT}_{ij}\right]}_{\text{dynamics! } 2 \text{ d.o.f. (polarizations)}}$$

$i \neq j \rightarrow \Phi = \Psi$ (non dynamical)

Linearised Einstein Equations

We can choose gauge $\partial_\mu h^\mu{}_\lambda - \frac{1}{2}\partial_\lambda h = 0$

the analogue of the Lorenz gauge in Maxwell theory $\partial_\mu A^\mu = 0$

As in Maxwell theory, this gauge choice does not necessarily fix the gauge completely.

Wave equation for graviton

In harmonic gauge the linearised Einstein equations are simply

$$\Box h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G_N T^{(0)}_{\mu\nu}$$

The vacuum equations (or the equations in a source-free region of spacetime) are just

$$T^{(0)}_{\mu\nu} = 0 \;\;\Rightarrow\;\; \Box h_{\mu\nu} = 0$$

It is convenient to consider the linear combination

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h$$

$$\boxed{\begin{aligned} \Box\bar{h}_{\mu\nu} &= -16\pi G_N T^{(0)}_{\mu\nu} \\ \partial_\mu \bar{h}^\mu_{\ \nu} &= 0 \ . \end{aligned}}$$

One of the solutions is the retarded solution

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G_N \int d^3x' \, \frac{T^{(0)}_{\mu\nu}(t - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}$$

Wave equation for graviton

$$\boxed{\begin{aligned} \Box \bar{h}_{\mu\nu} &= -16\pi G_N T^{(0)}_{\mu\nu} \\ \partial_\mu \bar{h}^\mu_{\ \nu} &= 0 \ . \end{aligned}}$$

One of the solutions is the retarded solution

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G_N \int d^3x' \, \frac{T^{(0)}_{\mu\nu}(t - |\vec{x}-\vec{x}'|, \vec{x}')}{|\vec{x}-\vec{x}'|}$$

This solution is automatically in the harmonic gauge because of $\partial_\mu T^{(0)\mu\nu} = 0$

The general solution is then a sum of this particular solution of the inhomogeneous equation and the general solution of the homogeneous equation

Polarization of gravitational waves

The linearised vacuum Einstein equation in the harmonic gauge,

$$\Box \bar{h}_{\mu\nu} = 0$$

Plane wave solution $\bar{h}_{\mu\nu} = \epsilon_{\mu\nu} \mathrm{e}^{\,ik_\alpha x^\alpha}$

where $\epsilon_{\mu\nu}$ is a constant, symmetric polarisation tensor and k^α is a constant wave null vector $k_\alpha k^\alpha = 0$.

The Einstein equations predict the existence of gravitational waves travelling along null geodesics, i.e. at the speed of light.

Too many parameters are to specify

Polarization of gravitational waves

$$\bar{h}_{\mu\nu} = \epsilon_{\mu\nu} \mathrm{e}^{ik_\alpha x^\alpha}$$

First of all, the harmonic gauge condition implies that

$$\partial_\mu \bar{h}^\mu_{\ \nu} = 0 \quad \Rightarrow \quad k^\mu \epsilon_{\mu\nu} = 0$$

Make use of the residual gauge freedom $\quad h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu$

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu - \eta_{\mu\nu} \partial_\lambda V^\lambda$$

$$\Rightarrow \quad \partial^\mu \bar{h}_{\mu\nu} \to \partial^\mu \bar{h}_{\mu\nu} + \Box V_\nu$$

The gauge condition is invariant precisely under transformations with $\Box V_\mu = 0$

Take the solution of the form $\quad V_\mu = v_\mu \mathrm{e}^{ik_\alpha x^\alpha}$

$$\epsilon_{\alpha\beta} \to \epsilon_{\alpha\beta} + i(k_\alpha v_\beta + k_\beta v_\alpha) - i\eta_{\alpha\beta} k^\gamma v_\gamma$$

We can choose the v_μ in such a way that the new polarisation tensor satisfies

$k^\mu \epsilon_{\mu\nu} = 0 \qquad \epsilon_{\mu 0} = \epsilon^\mu_{\ \mu} = 0 \qquad$ (8 independent conditions)

2 independent polarisations

Polarization of gravitational waves

In terms of $\bar{h}_{\mu\nu}$ we can write

$$k^\mu \bar{h}_{\mu\nu} = 0 \quad , \quad \bar{h}_{\mu 0} = 0 \quad , \quad \bar{h}^\mu_{\ \mu} = 0$$

Transverse traceless gauge $\bar{h}^{TT}_{\mu\nu}$

Example:

Consider a wave travelling in the x^3-direction, $k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega)$

It is easy to see that the only independent components are ϵ_{ab} with $a, b = 1, 2$.

ϵ_{ab} is symmetric and traceless $\Rightarrow \epsilon_{11} = -\epsilon_{22}$, $\epsilon_{12} = \epsilon_{21}$.

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_1 & C_2 & 0 \\ 0 & C_2 & -C_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The solution for the metric itself: $ds^2 = -dt^2 + (\delta_{ab} + h_{ab})dx^a dx^b + (dx^3)^2$

with $h_{ab} = h_{ab}(t \mp x^3)$ symmetric and traceless.

Effects of a gravitational wave

Influence of Curvature on Particle Trajectories (gravitational tidal forces):
How gravitational tidal forces change trajectories of nearby particles ?

the geodesic equation for x^μ $$\frac{d^2}{d\tau^2}x^\mu + \Gamma^\mu_{\ \nu\lambda}(x)\frac{d}{d\tau}x^\nu \frac{d}{d\tau}x^\lambda = 0$$

and for $$\frac{d^2}{d\tau^2}(x^\mu + \delta x^\mu) + \Gamma^\mu_{\ \nu\lambda}(x+\delta x)\frac{d}{d\tau}(x^\nu + \delta x^\nu)\frac{d}{d\tau}(x^\lambda + \delta x^\lambda) = 0$$

Combining the two equations one can find *geodesic deviation equation*

$$(D_\tau)^2 \delta x^\mu = R^\mu_{\ \nu\lambda\rho}\, u^\nu u^\rho\, \delta x^\rho$$

With the covariant operator $$D_\tau \delta x^\mu = \frac{d}{d\tau}\delta x^\mu + \Gamma^\mu_{\ \nu\lambda}\frac{dx^\nu}{d\tau}\delta x^\lambda$$

Effects of a gravitational wave

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$$(D_\tau)^2 S^\mu = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho S^\sigma$$

Let the test particles are initially at rest $u^\mu = (1, 0, 0, 0)$

When a gravitational wave arrives $u^\mu = (1, 0, 0, 0) + \mathcal{O}(h)$

The Riemann tensor is already of order $h \Rightarrow$ we have at the lowest order

$$R^{(1)}_{\mu 00\sigma} = \frac{1}{2}\partial_0\partial_0 h_{\mu\sigma}$$

The geodesic deviation equation becomes

$$\boxed{\ddot{S}^\mu = \frac{1}{2}\ddot{h}^\mu{}_\sigma\, S^\sigma}$$

Effects of a gravitational wave

$$\ddot{S}^{\mu} = \frac{1}{2}\ddot{h}^{\mu}_{\ \sigma}\, S^{\sigma}$$

$S^3 = 0$ for a wave propagating in x^3 direction $\Rightarrow$ the gravitational wave is *transversally polarised*. The particles are only disturbed in directions perpendicular to the wave.

The movement of the particles in the 1-2 plane is then governed by

$$\ddot{S}^{a} = \frac{1}{2}\ddot{h}^{a}_{\ b}\, S^{b} \equiv -(\Omega^2)^{a}_{\ b}\, S^{b}$$

$$h^{a}_{\ b} = \mathrm{e}^{-i\omega(t - x^3)}\epsilon^{a}_{\ b} \qquad \Longrightarrow \qquad (\Omega^2)^{a}_{\ b} = \frac{1}{2}\omega^2 h^{a}_{\ b}$$

we can consider separately the two cases (1) $\epsilon_{12} = 0$ and (2) $\epsilon_{11} = -\epsilon_{22} = 0$.

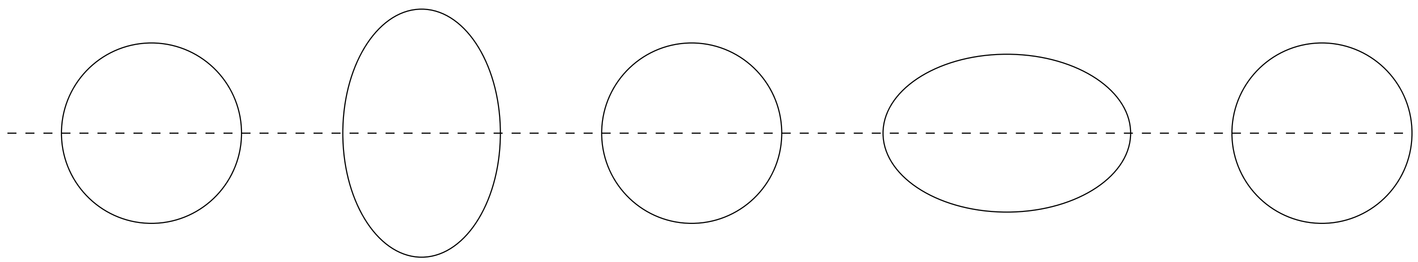
Effects of a gravitational wave

For $\epsilon_{12} = 0$ one has

$$\ddot{S}^1(t) = -\tfrac{1}{2}\epsilon_{11}\omega^2 \mathrm{e}^{-i\omega t} S^1(t)\, \mathrm{e}^{i\omega x^3}$$
$$\ddot{S}^2(t) = +\tfrac{1}{2}\epsilon_{11}\omega^2 \mathrm{e}^{-i\omega t} S^2(t)\, \mathrm{e}^{i\omega x^3}$$

the solution to lowest order (in ϵ)

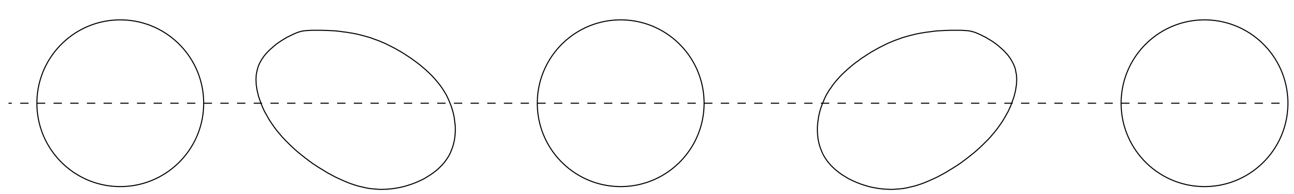
$$S^1(t) = (1 + \tfrac{1}{2}\epsilon_{11}\mathrm{e}^{-i\omega(t - x^3)})S^1(0)$$
$$S^2(t) = (1 - \tfrac{1}{2}\epsilon_{11}\mathrm{e}^{-i\omega(t - x^3)})S^2(0)$$



$+$ polarisation

$\epsilon_{11} = 0$ but $\epsilon_{12} = \epsilon_{21} \neq 0$,

$$S^1(t) = S^1(0) + \tfrac{1}{2}\epsilon_{12}\mathrm{e}^{-i\omega(t - x^3)}S^2(0)$$
$$S^2(t) = S^2(0) + \tfrac{1}{2}\epsilon_{12}\mathrm{e}^{-i\omega(t - x^3)}S^1(0)$$



$\times$ polarisation

Production Gravitational Waves

we need to include sources

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G_N \int d^3y \, \frac{T^{(0)}_{\mu\nu}(t-|\vec{x}-\vec{y}|,\vec{y})}{|\vec{x}-\vec{y}|}$$

At large distances, and when the wavelength is much larger than the size of the source

$$\bar{h}_{\mu\nu}(t,\vec{x}) \approx \frac{4G_N}{r} \int d^3y \, T^{ret}_{\mu\nu}(t,\vec{y})$$

$$r = |\vec{x}|$$

$$T^{ret}_{\mu\nu}(t,\vec{y}) = T^{(0)}_{\mu\nu}(t-r,\vec{y})$$

The gravitational analogue of the dipole approximation in electrodynamics

In this approximation the leading $(1/r)$- part of $h_{\mu 0}$ does not lead to gravitational waves

Production Gravitational Waves

concentrate on the spatial components

$$\bar{h}_{ik}(t,\vec{x}) \approx \frac{4G_N}{r}\int d^3y\, T^{ret}_{ik}(t,\vec{y})$$

$$\bar{h}_{ik}(t,\vec{x}) \approx \frac{2G_N}{r}\ddot{Q}^{ret}_{ik} \qquad\qquad Q^{ret}_{ik}(t) = \int d^3x\, \rho^{ret} x_i x_k$$

In case when $\rho(t) \sim \mathrm{e}^{-i\Omega t}$ $\Rightarrow$ $\bar{h}_{ik}(t,r) \approx -2G_N\Omega^2 Q^{ret}_{ik}\,\dfrac{\mathrm{e}^{-i\Omega(t-r)}}{r}$

an outgoing spherical wave.

$$\boxed{\frac{dE}{dt} = -\frac{G_N}{5}(\dddot{\mathcal{Q}}^{ret})_{ik}(\dddot{\mathcal{Q}}^{ret})^{ik}}$$

$$\begin{aligned}\mathcal{Q}^{ret}_{ik} &= \int d^3x\, \rho^{ret}(x_i x_k - \tfrac{1}{3}\delta_{ik} r^2) \\ &= Q^{ret}_{ik} - \tfrac{1}{3}\delta_{ik}(Q^{ret})^j_{\ j} \quad .\end{aligned}$$