Simple Models of Gravitational Collapse

Collapse of a Shell of Radiation

Consider an infinitely thin spherical shell of radiation in an otherwise empty spacetime.

Minkowski spacetime

$$ds^2 = -dv^2 + 2dv\ dr + r^2 d\Omega^2$$

Schwarzschild metric (with $v = t + r_*$) has the ingoing Eddington-Finkelstein form

$$ds^2 = -f(r)dv^2 + 2dv\ dr + r^2 d\Omega^2 \quad , \quad f(r) = 1 - \frac{2m}{r}$$

In both metrics, ingoing lightrays are described by lines of constant v.

- the flat Minkowski geometry inside the shell
- and the Schwarzschild metric outside the shell.

The shell moves along the ingoing null trajectory $v = v_0$, as viewed from both the internal Minkowski geometry and the external Schwarzschild geometry.

Simple Models of Gravitational Collapse

$$ds^2 = -f(v,r)dv^2 + 2dv\ dr + r^2 d\Omega^2 \quad , \quad f(v,r) = 1 - \frac{2m_f}{r}\Theta(v - v_0)$$

The form of ingoing Vaidya metric:

$$ds^2 = -f(v,r)dv^2 + 2dv\ dr + r^2 d\Omega^2 \quad , \quad f(v,r) = 1 - \frac{2m(v)}{r}$$

$$m(v) = m_f \Theta(v - v_0)$$

Einstein equations are satisfied by choosing

$$T_{vv} = \frac{1}{4\pi G_N}\frac{m_f}{r^2}\delta(v - v_0)$$ Purely ingoing light

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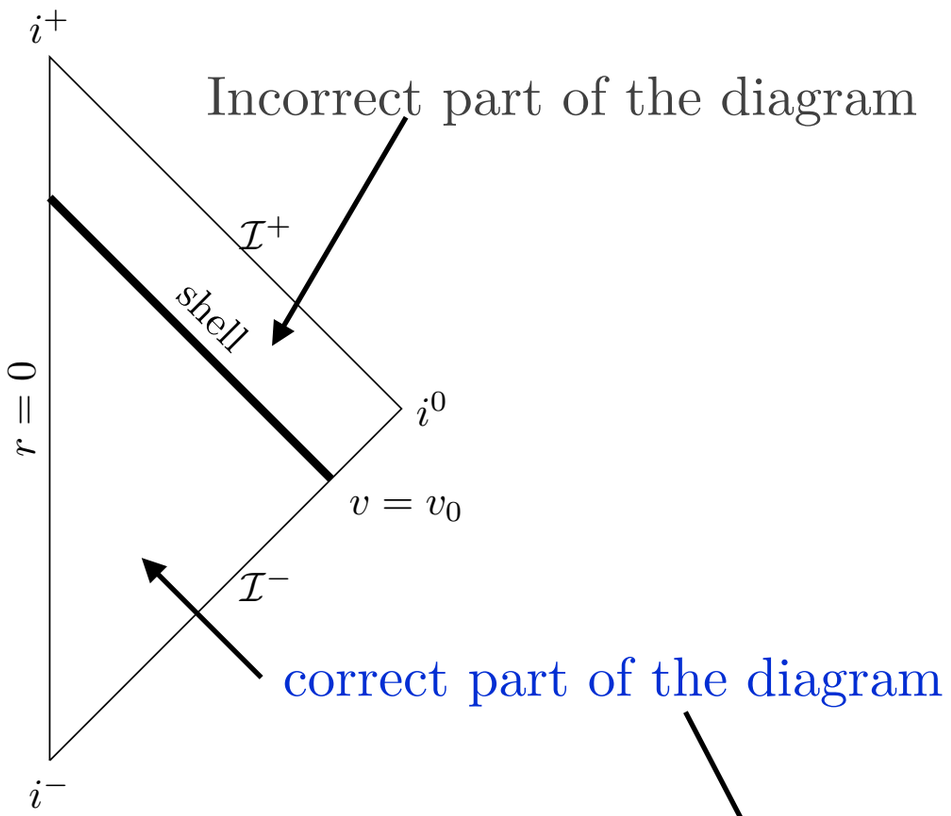
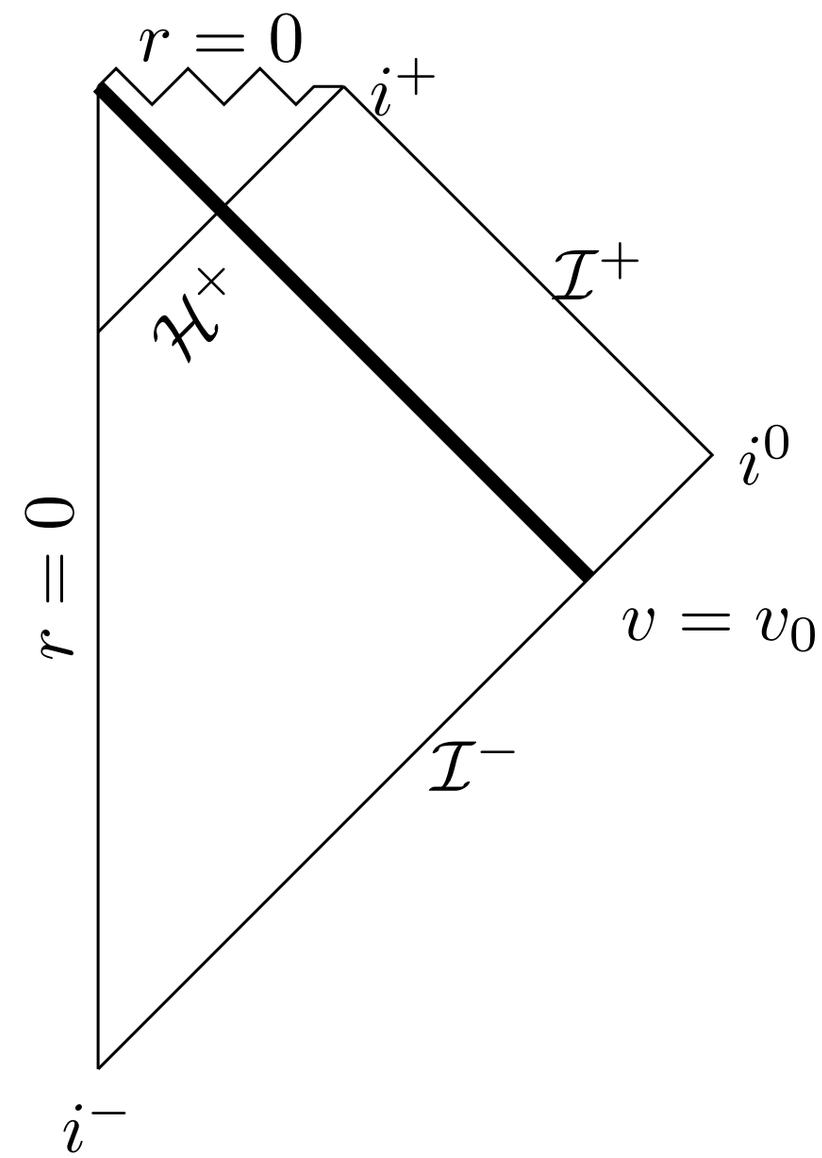
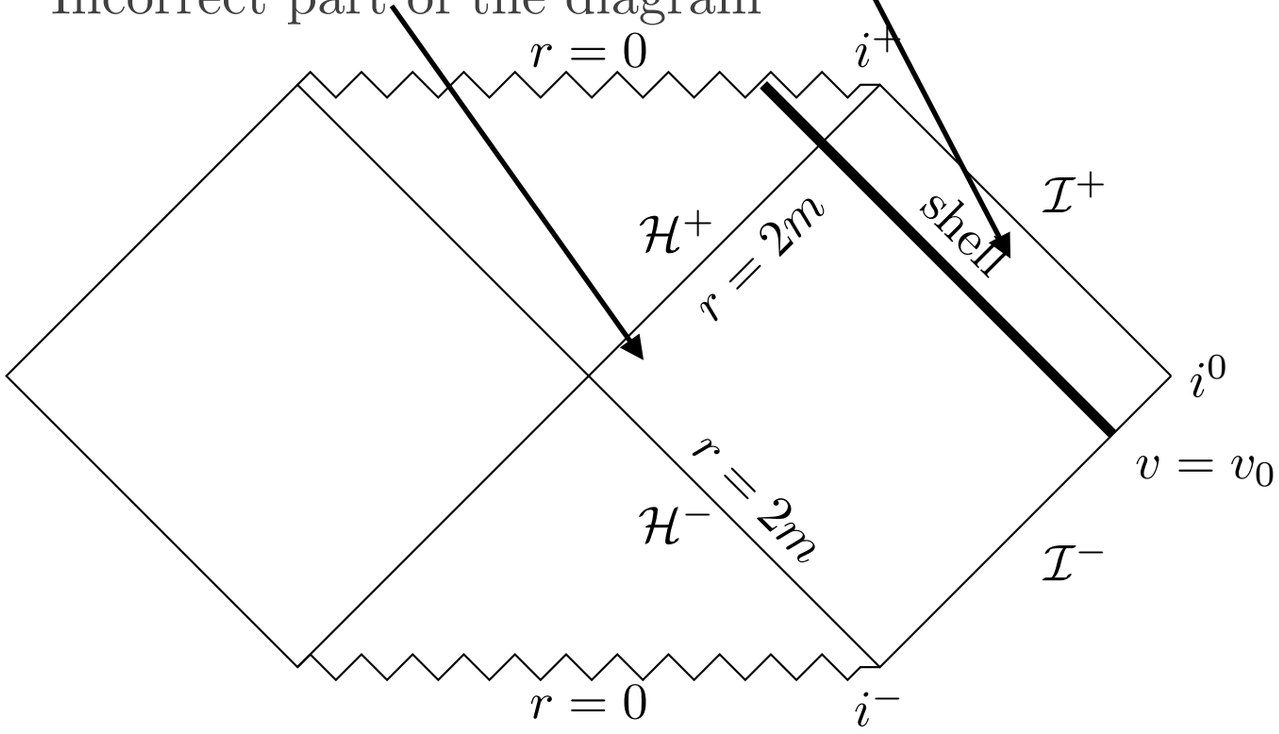


correct diagram



correct part of the diagram

Incorrect part of the diagram



Rotating black holes

Kerr metric

$$ds^2 = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 - \frac{4mra\sin^2\theta}{\rho^2} dt\, d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

$$\Delta(r) = r^2 - 2mr + a^2$$
$$\rho(r,\theta)^2 = r^2 + a^2\cos^2\theta$$
$$\Sigma(r,\theta) = (r^2 + a^2)^2 - \Delta(r)a^2\sin^2\theta$$

- Two free parameters m and a (rotation parameter), $J = Ma$.
- This metric is time-independent and axially symmetric, with the two commuting Killing vectors $\xi = \partial_t$, $\eta = \partial_\phi$.
- Stationary (but not static) $\Rightarrow$ there is a term $g_{t\phi}$, no symmetry $t \to -t$ (More invariantly, this is the statement that the Killing vector t is not hypersurface-orthogonal)
- It reduces to the Schwarzschild metric for $a = 0$. Also reduces to the Minkowski spacetime when $m = 0$, but in 'weird' rotating coordinates.
- Electric charge can be added to this solution by the same replacement $m \to m - Q^2/(2r)$.

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- A symmetry $\theta \to \theta - \pi$. Also a symmetry $(t, \phi) \to (-t, -\phi)$, related to *circularity* property of the solution. (Running backwards in time negative spin = running forward in time with positive spin.)

$$\xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(t)} = \xi_{(t)} \wedge \xi_{(\varphi)} \wedge \mathrm{d}\xi_{(\varphi)} = 0\ .$$

- Asymptotically

$$ds^2 \simeq -\left(1 - \frac{2m}{r}\right) dt^2 - \frac{4ma\sin^2\theta}{r} dt d\phi + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Kerr black hole: ergosphere

$$ds^2 = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 - \frac{4mra\sin^2\theta}{\rho^2} dt\, d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

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- Static observers, remaining at fixed values of the spatial coordinates (r, θ, ϕ), with 4-velocity

$$u^\alpha \sim \xi^\alpha$$

- A static limit or infinite redshift surface for such observers?

$$\xi^2 = g_{tt} = 0$$

$$g_{tt}(r,\theta) = 0 \quad \Leftrightarrow \quad \rho^2 - 2mr = r^2 + a^2\cos^2\theta - 2mr = 0$$

$$r_{sl}(\theta) = m + \sqrt{m^2 - a^2\cos^2\theta}$$

Kerr black hole: ergosphere

- No curvature singularity at $r = r_{sl}$.
- It defines the static limit surface for static observers: no static observers can exist for $r < r_{sl}$
- Also defines a surface of infinite redshift for static observers.
- Killing vector $\xi = \partial_t$ becomes null.
- For the Schwarzschild metric, $r_{sl} \to 2m = r_g$ reduces to the Schwarzschild radius.

However this surface is not a horizon

Kerr black hole: ergosphere

- Although no static observers can exist for $r < r_{sl}$ this does not by itself imply that one cannot escape from that region. Stationary observers can escape with $u^\alpha \sim \xi^\alpha + \Omega\eta^\alpha$.
- This surface is timelike, it has a spacelike normal.

$$S(r,\theta) = r - r_{sl}(\theta) = 0$$

Normal vector $\qquad N_\alpha = \partial_\alpha S : \quad N_\alpha = (0, 1, -dr_{sl}/d\theta, 0)$

$$N_\alpha N^\alpha = g^{rr} + g^{\theta\theta}(dr_{sl}(\theta)/d\theta)^2 = \frac{1}{2mr_{sl}} \frac{m^2 a^2 \sin^2\theta}{m^2 - a^2\cos^2\theta} \geq 0$$

Clearly cannot be a horizon

Kerr black hole: inside the ergosphere

- Stationary observers $u^\alpha \sim \xi^\alpha + \Omega\eta^\alpha$.
- u^α is timelike when $g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} < 0$

$$\Omega_-(r,\theta) < \Omega(r,\theta) < \Omega_+(r,\theta)$$

$$\Omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}}$$

$$\Omega_- \quad \begin{cases} < 0 & \text{for} \quad r > r_{sl} \\ = 0 & \text{for} \quad r = r_{sl} \\ > 0 & \text{for} \quad r < r_{sl} \end{cases}$$

- Outside the ergosphere stationary observers can rotate either with or against the rotation of the black hole. On and inside the ergosphere a stationary observer only rotate with (i.e. to be dragged along by) the black hole.
- One can add a small motion in the direction of r or θ and see that one can escape from the inside of the ergosphere, as long as stationary observers exist.

Kerr black hole: Horizon

- Stationary observers cease to exist when

$$g_{t\phi}^2 - g_{tt}g_{\phi\phi} = \Delta(r)\sin^2\theta \ = 0 \qquad \text{also} \quad g^{rr} = \frac{\Delta}{\rho^2}$$

$$\Delta(r) = r^2 - 2mr + a^2 = 0 \qquad \Rightarrow \qquad r_\pm = m \pm \sqrt{m^2 - a^2}$$

- We note that $r_+ = m + \sqrt{m^2 - a^2} \leq m + \sqrt{m^2 - a^2\cos^2\theta} = r_{sl}(\theta)$

  Equality is only at the poles $\theta = 0, \pi$

- Hypersurface of constant r with the normal $N_\alpha \sim \partial_\alpha r$ becomes null at r_+:

$$g^{\alpha\beta}\partial_\alpha r \partial_\beta r = 0$$

Kerr black hole: Horizon

- At $r = r_+$ the stationary observers have the angular velocity

$$\Omega_h \equiv \omega(r_+) = \frac{a}{r_+^2 + a^2}$$

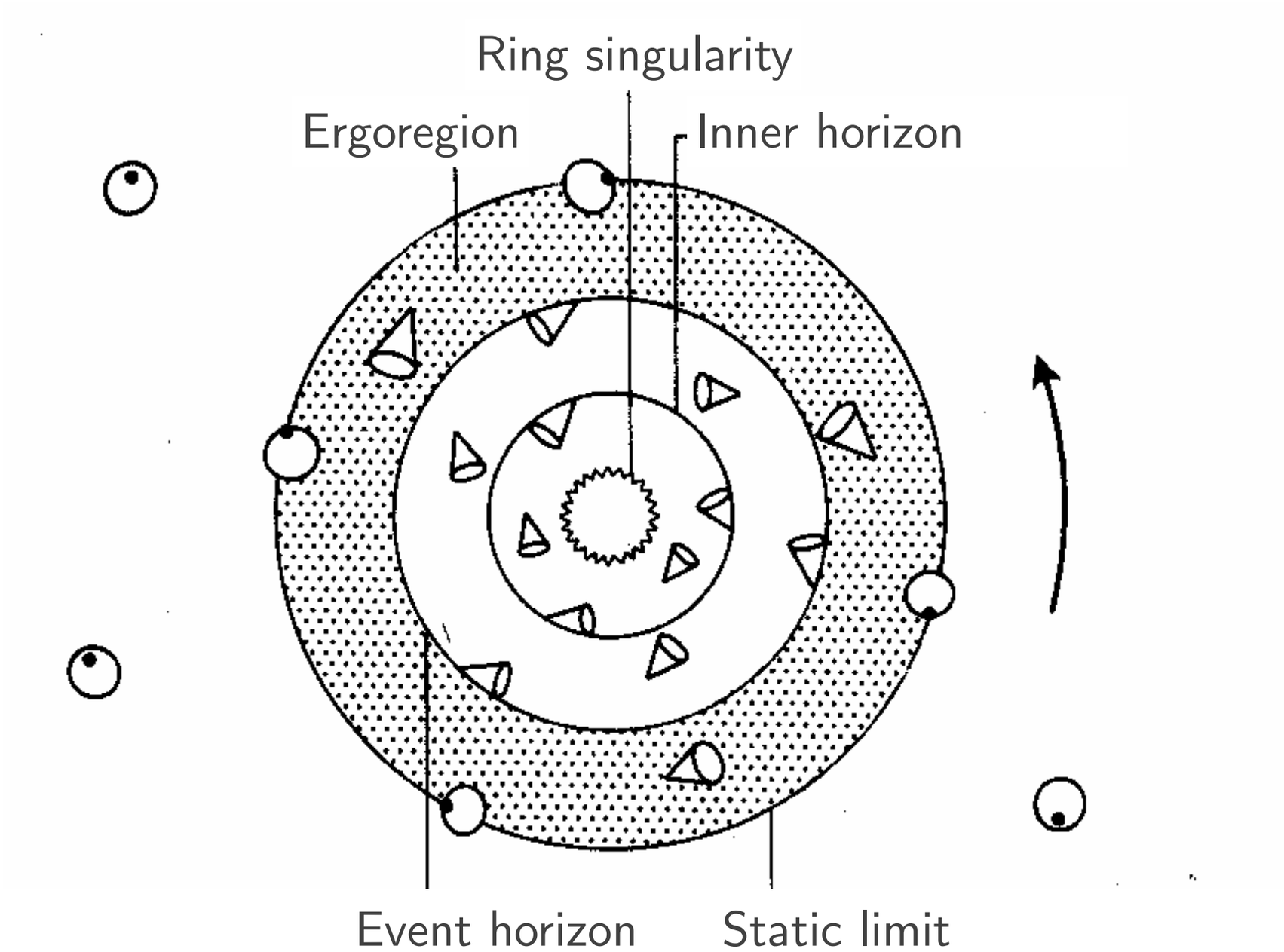
- The Killing vector corresponding to the stationary observers there is null

$$\xi_h \equiv \xi_{\Omega_h} = \xi + \Omega_h \eta \qquad \Rightarrow \qquad \left(g_{\alpha\beta} \xi_h^\alpha \xi_h^\beta \right) |_{r=r_+} = 0$$

- This surface is a Killing horizon and also the event horizon

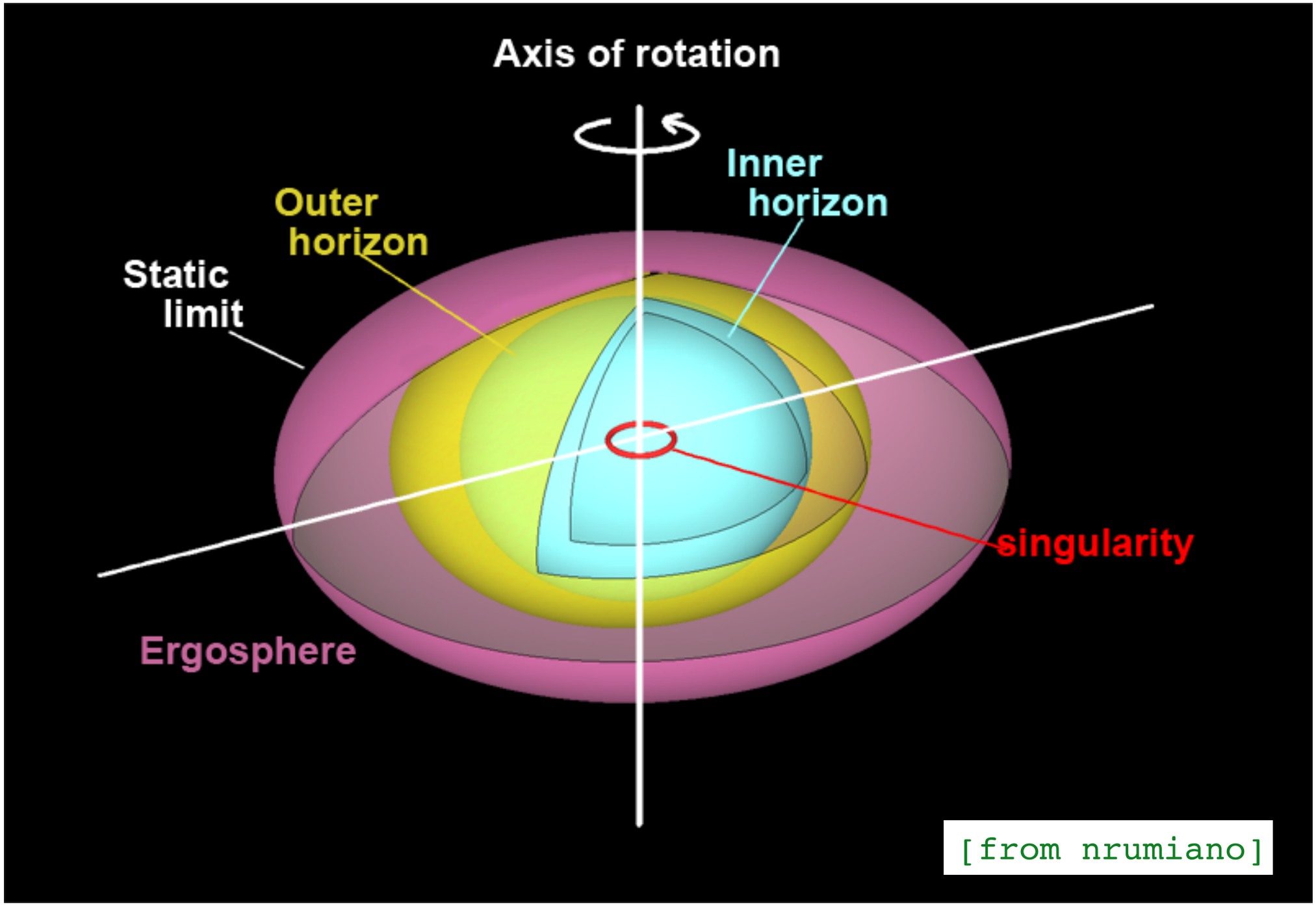
- Singularity occurs at $\rho^2 = r^2 + a^2 \cos^2\theta = 0 \Rightarrow r = \cos\theta = 0$ It is a ring singularity, $x^2 + y^2 = a^2$, $z = 0$.

Important surfaces in the Kerr metric



[from d'Inverno's book]

Kerr summary



Killing horizon vs event horizon

- Rigidity theoreom (Hawking): The event horizon $\mathcal{H}$ of a real analytic, stationary, regular, vacuum spacetime is a Killing horizon: $\exists$ a Killing field k normal to $\mathcal{H}$ which verifies $k^2 = 0$ on $\mathcal{H}$.

- For the outer horizon of the Kerr spacetime, this Killing vector is

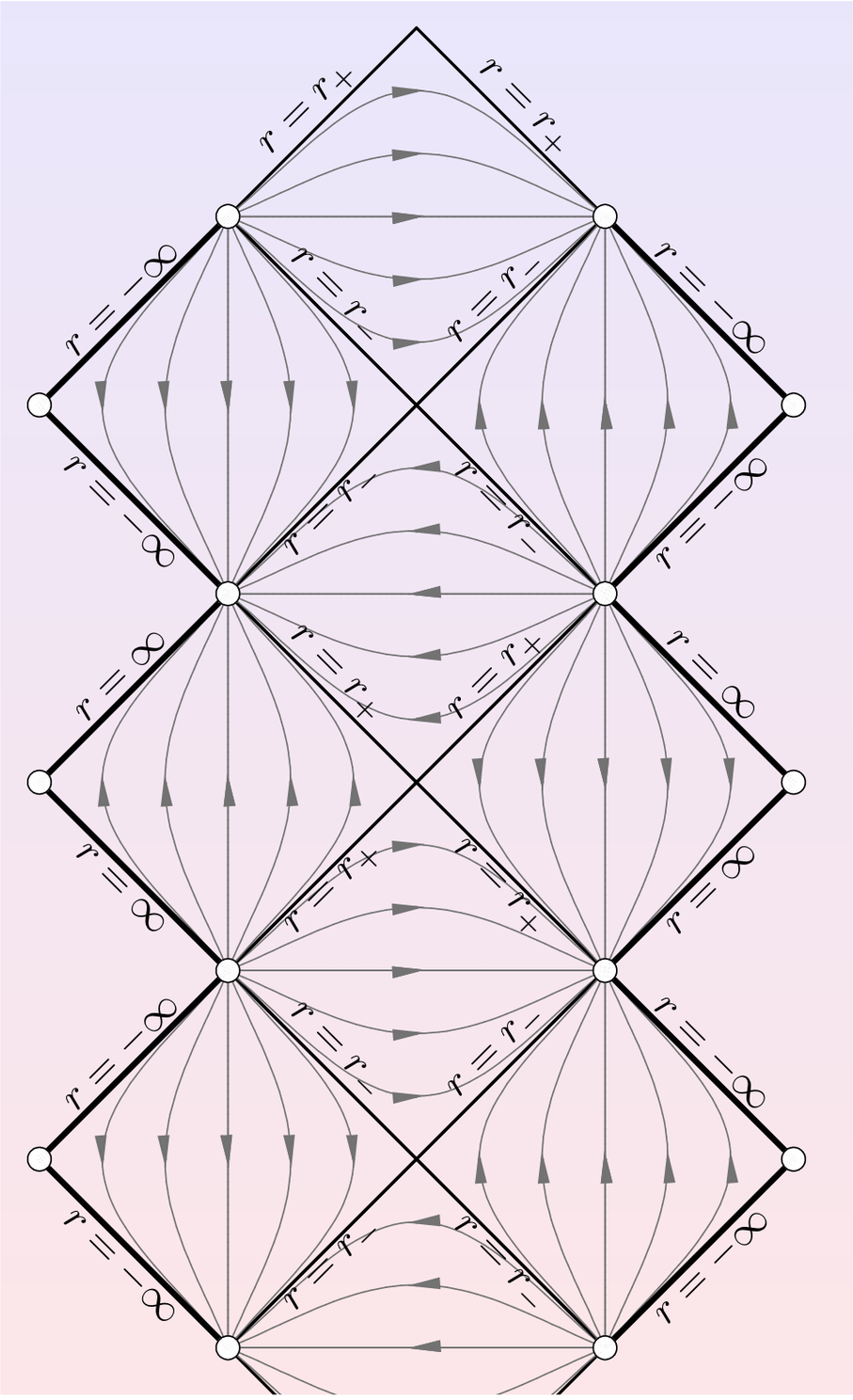
$$k = \partial_t + \frac{a}{2Mr_+}\partial_\varphi$$

- One can define the surface gravity κ_+ of $\mathcal{H}$ as

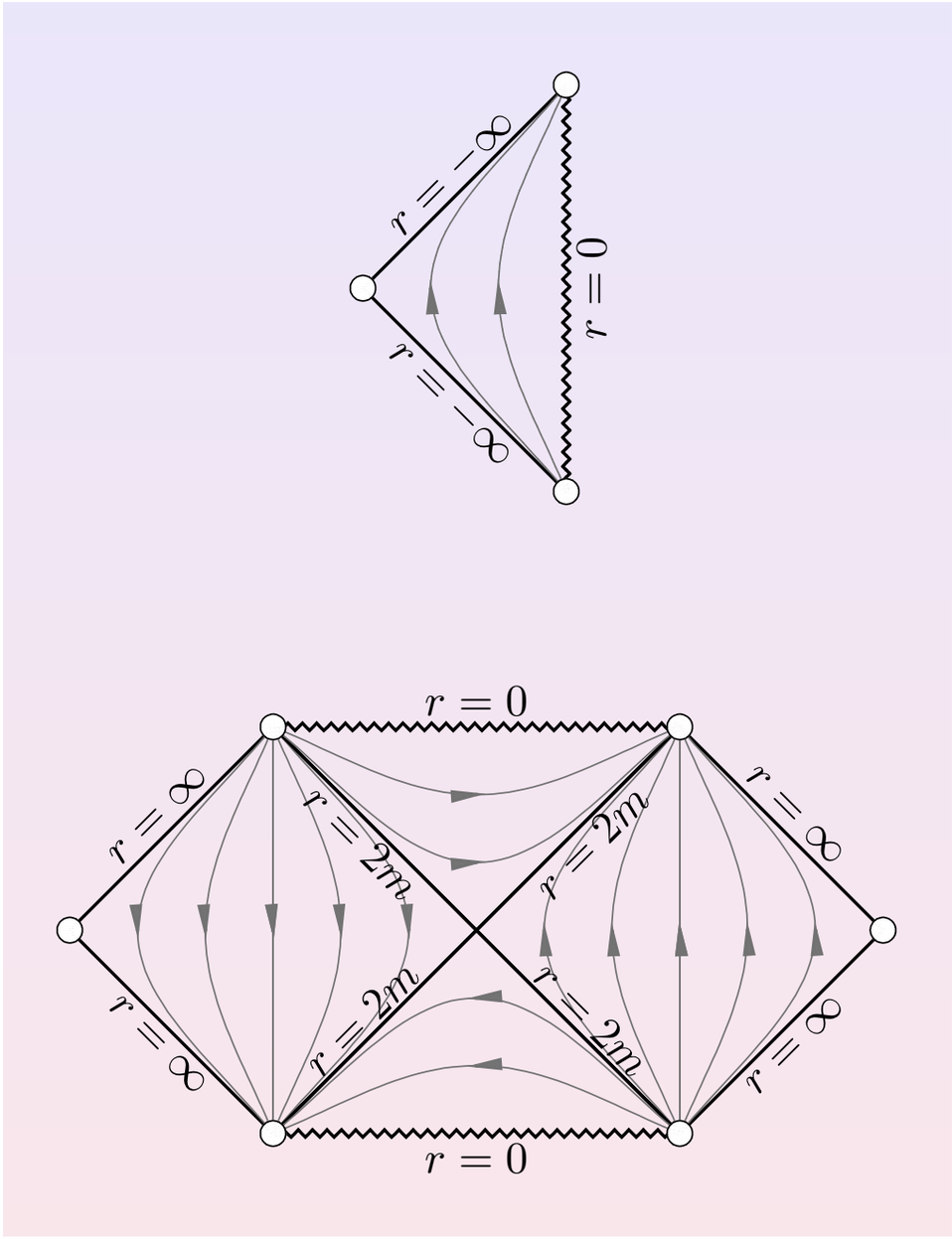
$$k^\mu \nabla_\mu k^\nu = \kappa_+ k^\nu$$

- The surface gravity is constant on $\mathcal{H}$ and is related to the Hawking temperature $T_H = \kappa_+/2\pi$

Conformal diagram for Kerr spacetime



Conformal diagram for $\theta = 0$



Conformal diagram for $\theta = \pi/2$

From a talk by Piotr T. Chrusciel

Kerr singularity, extremal black holes, time machines

- For $a > M$ there is no horizon $\Rightarrow$ naked singularity
- Extremal black hole $a = M$
- Time machine inside Kerr black hole

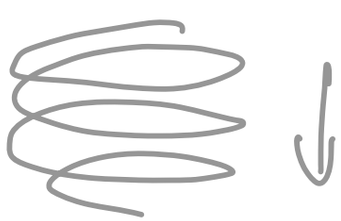
$$ds^2 = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 - \frac{4mra\sin^2\theta}{\rho^2} dt\, d\phi + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

$$\Sigma = \left(r^2 + a^2\right)^2 - \Delta(r) a^2 \sin^2\theta = \left(r^2 + a^2\right)\rho^2 + 2mra^2\sin^2\theta$$

Consider ρ close to 0, i.e. close to the ring singularity.

There is a region $(r < 0)$ where the sign changes.

$\partial/\partial\phi$ is timelike. Consider now $t = const$, $r = const$, $\theta = const$, and motion along $\phi-$direction $\Rightarrow$ closed timelike curves.

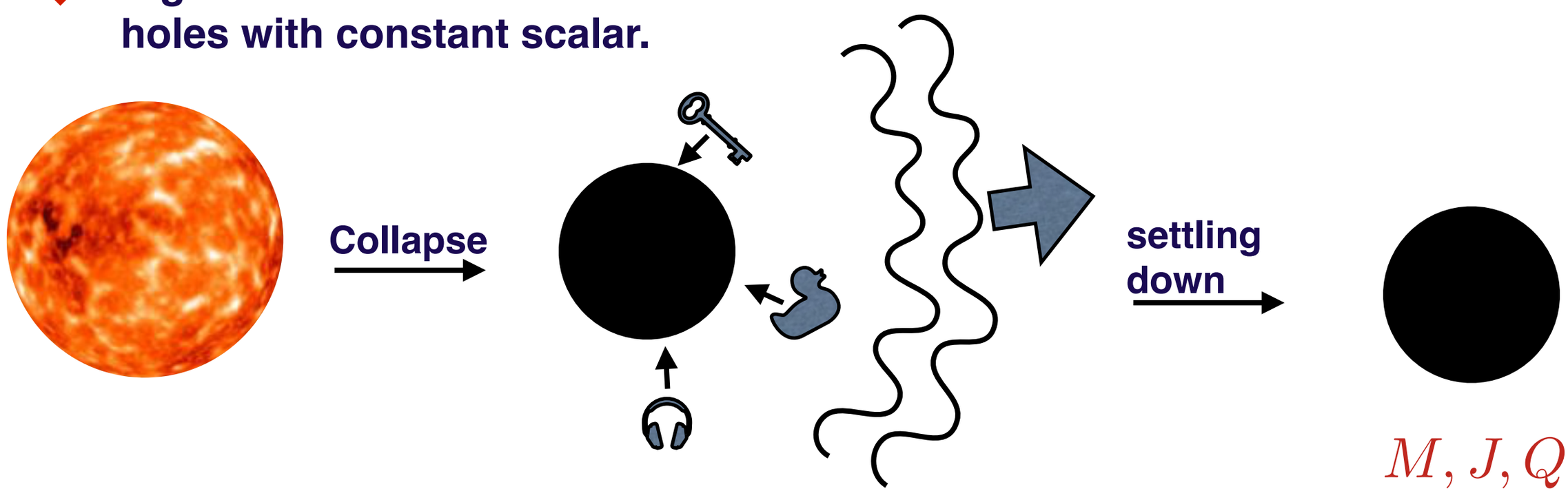


Decreasing $t \Rightarrow$ time machine

Black holes are bald

- Gravitational collapse...
- Black holes eat or expel surrounding matter
- Their stationary phase is characterised by a limited number of charges
- No details about collapse
- Black holes are bald

- **No hair theorems/arguments dictate that adding degrees of freedom lead to trivial (General Relativity) or singular solutions.**
- **E.g. in the standard scalar-tensor theories BH solutions are GR black holes with constant scalar.**



Example of hairy black hole

BBMB solution

Bocharova et al'70, Bekenstein'74

Conformally coupled scalar field:

$$S[g_{\mu\nu},\phi] = \int \sqrt{-g}\left(\frac{R}{16\pi G} - \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - \frac{1}{12}R\phi^2\right)d^4x$$

Static spherically symmetric (nontrivial) solution:

$$ds^2 = -\left(1-\frac{m}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1-\frac{m}{r}\right)^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Secondary scalar hair:

$$\phi = \sqrt{\frac{3}{4\pi G}}\frac{m}{r-m}$$

NB. The geometry is of that of extremal RN.
The scalar field is unbounded at r=m

Gauss-Bonnet term

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \lambda\phi\hat{G}\right]$$

Gauss-Bonnet invariant: $\hat{G} = R_{\mu\nu\sigma\alpha}R^{\mu\nu\sigma\alpha} - 4R_{\mu\nu}R^{\mu\nu} + R^2$

Horndeski theory with $G_5 \propto \ln|X| \Rightarrow$ assumption (iii) is violated.

EoM for the scalar: $\Box\phi = -\lambda\hat{G}$

Source for the scalar: it cannot be trivial in BH background

Campbell et al'92
Kanti et al'96
Sotiriou and Zhou'13

Solutions in "Galileon theory"

EB, Charmousis'13

$$\mathcal{L}^{\Lambda\mathrm{CGJ}} = R - \eta(\partial\phi)^2 + \beta G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2\Lambda.$$

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$$\phi = qt + \psi(r)$$

Time-dependent scalar !

Asymptotically dS/AdS:

$$f = h = 1 - \frac{\mu}{r} - \frac{\Lambda_{\mathrm{eff}}}{3}r^2, \qquad \psi' = \pm\frac{q}{h}\sqrt{1-h}, \qquad \Lambda_{\mathrm{eff}} = -\frac{1}{2\beta}$$

Properties of disformed Kerr in modified gravity

EB, T.Anson, C.Charmousis, M.Hassaine'20

- *Ergosphere* (static limit): static timelike observers can no longer exist, the Killing vector $l^\mu = (1,0,0,0)$ becomes null. I.e. $\tilde{g}_{tt} = 0$, or

$$\tilde{g}_{tt} = 0 \quad \Rightarrow \quad r_E = \tilde{M} + \sqrt{\tilde{M}^2 - a^2\cos^2\theta}$$

- *Stationary limit.* Observers constant (r,θ), with a 4-velocity $u = \partial_t + \omega\partial_\varphi$. These observers cease to exist at the surface $\tilde{g}_{tt}\tilde{g}_{\varphi\varphi} - \tilde{g}_{t\varphi}^2 = 0$, i.e.

$$P(r,\theta) \equiv r^2 + a^2 - 2\tilde{M}r + \frac{2\tilde{M}Da^2 r\sin^2\theta}{\rho^2(r,\theta)} = 0$$

The surface is *timelike* and thus cannot be a horizon.

- *Horizon*: a null hypersurface of the form $r = R(\theta)$. The normal has components

$$n_\mu = (0, 1, -R'(\theta), 0)$$

The condition $n^2 = 0$ yields

$$R'(\theta)^2 + P(R,\theta) = R'(\theta)^2 + R^2 + a^2 - 2\tilde{M}R + \frac{2\tilde{M}Da^2 R\sin^2\theta}{\rho^2(R,\theta)} = 0$$

Surfaces