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Black holes and gravitational waves

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Outline

- Lecture 1: Schwarzschild black holes in General Relativity
- Lecture 2: Rotating black holes
- Lecture 3: Gravitational waves
- Lecture 4: Black holes and gravitational waves, observations

Schwarzschild black holes

General Relativity

- General Relativity is a field theory based on a single metric $g_{\mu\nu}$ (no other metrics or other fields) $\Leftarrow$ encodes curvature of space time
- line element:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- metric connection

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$$

- geodesics

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

- Riemann tensor

$$R^\alpha{}_{\beta\gamma\delta} \equiv \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\mu\gamma} \Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\mu\delta} \Gamma^\mu_{\beta\gamma}$$

General Relativity

- Ricci tensor

$$R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$$

- Ricci scalar

$$R = R^{\alpha}{}_{\alpha}$$

- Action for gravity (Einstein-Hilbert action):

$$S_{EH} = \frac{M_{Pl}^2}{2} \int d^4x \sqrt{-g} R + S_m[g, \Psi_m]$$

$$M_{Pl}^2 = \frac{1}{8\pi G}$$

- Variation with respect to $g_{\mu\nu}$ $\quad \rightarrow \quad$ Einstein equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$$

Vacuum solutions in General Relativity

No matter in spacetime (part of spacetime)

$$T_{\mu\nu} = 0 \quad \to \quad G_{\mu\nu} = 0$$

Spherically symmetric solutions

- One can show that for spherically symmetric solutions:

  $$ds^2 = -Adt^2 + 2Bdtdr + Cdr^2 + D\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

  A, B, C, D depend only on t and r.

- Change of coords: $D \to r^2$, $B \to 0$

  $$ds^2 = -e^{-\nu}dt^2 + e^{\lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Schwarzschild solution

$$G_{\mu\nu} = 0 \quad \rightarrow \begin{cases} e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \\ e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0 \\ \dot{\lambda} = 0 \end{cases}$$

(independent equations)

- (3) $\rightarrow \quad \lambda = \lambda(r)$
- (1) is ODE, and it can be integrated to give $e^{\lambda} = \frac{1}{1 - r_g}r$
- (1)+(2) $\rightarrow \lambda' + \nu' = 0 \quad \rightarrow \lambda + \nu = h(t)$, $\nu = -\lambda(r) + h(t)$
-

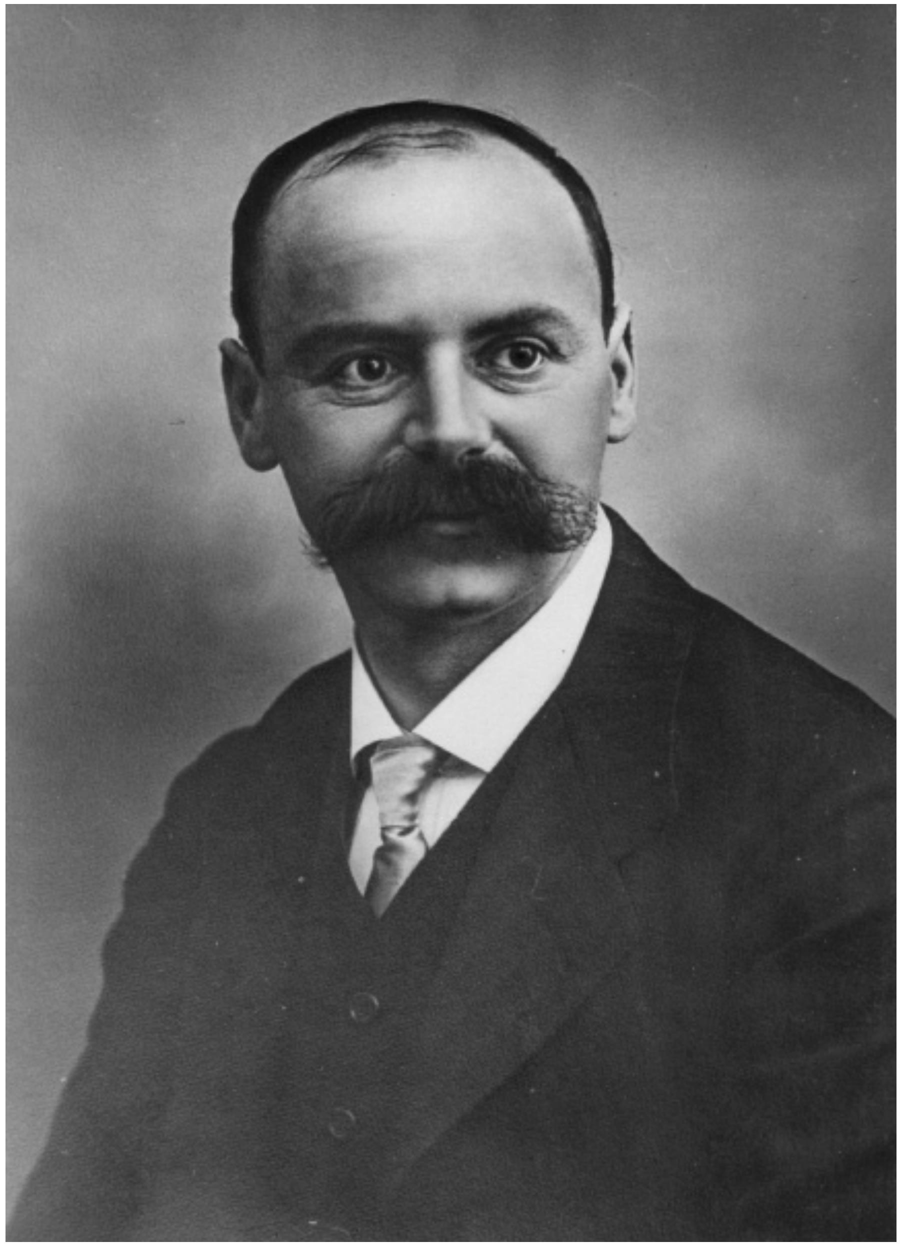
$$ds^2 = -e^{h(t)}\left(1 - \frac{r_g}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

$e^{h(t)}dt^2 \rightarrow dt^2$

$$\boxed{ds^2 = -e^{h(t)}\left(1 - \frac{r_g}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)}$$

Schwarzschild solution

Karl Schwarzschild



The solution was found in 1916

Schwarzschild solution

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- The solution is asymptotically flat, that is at $r \to 0$

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

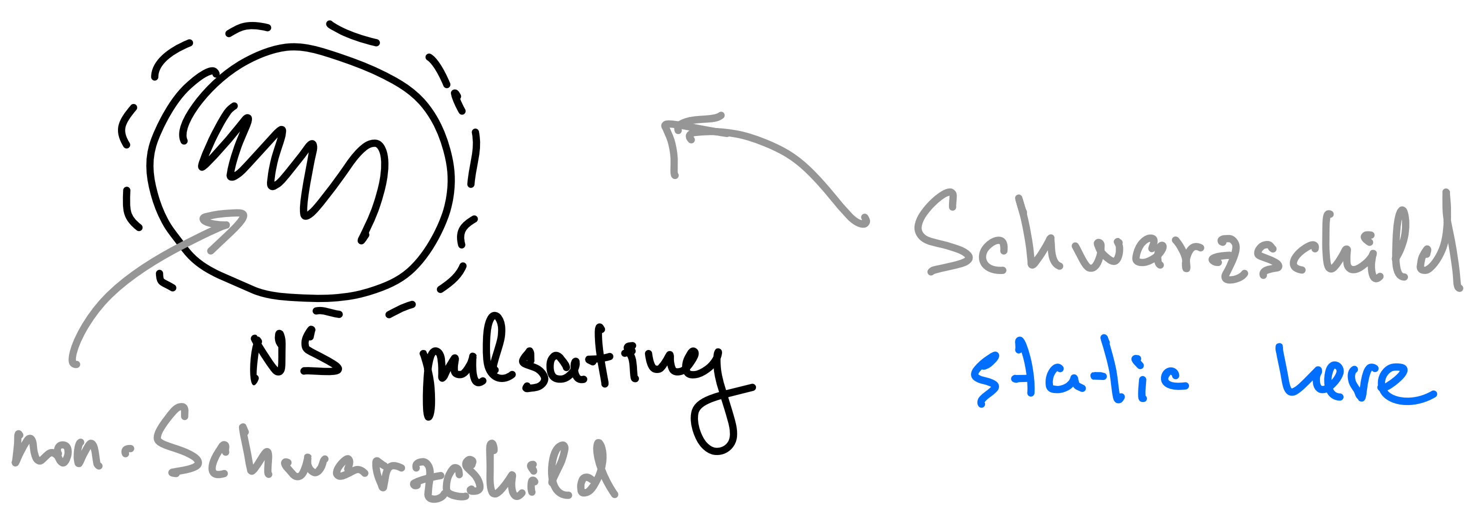
- Static, meaning that the Killing vector ∂_t is hypersurface orthogonal (There is a coordinate system such that $g_{t\alpha} = 0$)

<u>Birkhoff's theorem (1923):</u> any spherically symmetric solution of the vacuum field equations must be (a piece) of the Schwarzschild solution.

Schwarzschild solution

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Any spherical perturbation of a black hole is a pure gauge (non-physical)
- For a massive body, there is no effect on the spacetime in case of spherically symmetric pulsations.



- $r_g = 2GM$, M is the mass of the black hole

Killing vectors

Killing vector (field) reflects a symmetry of the metric

$$x^\mu \to y^\mu(x^\nu) \qquad \Longrightarrow \qquad g'_{\mu\nu}(y(x)) = \frac{\partial x^\rho}{\partial y^\mu}\frac{\partial x^\lambda}{\partial y^\nu} g_{\rho\lambda}(x)$$

Compare the new and old metrics in *different* points of spacetime P and P', but with the same values of coordinates $y^\mu(P') = x^\mu(P)$

A symmetry of the metric: $\quad g'_{\mu\nu}(y) = g_{\mu\nu}(y)$

The metric does not change along a Killing vector V^μ: $\quad L_V\, g_{\mu\nu} = 0$

$$\Updownarrow$$

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$$

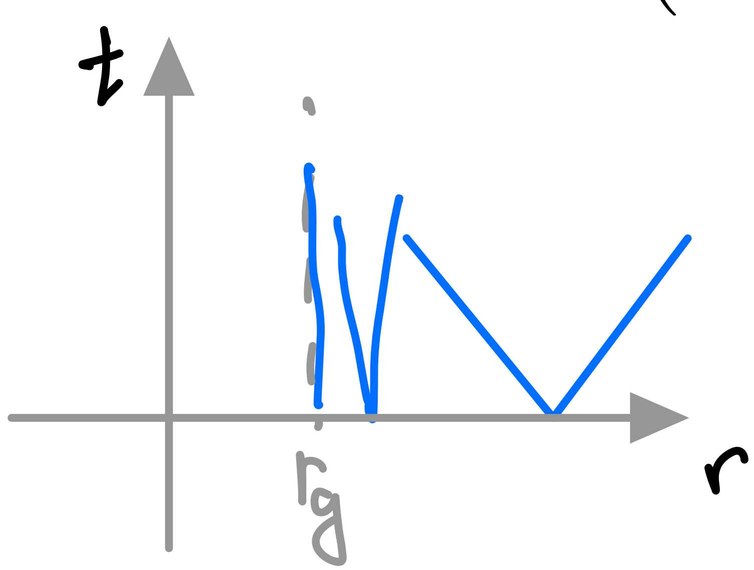
Schwarzschild solution: singularity?

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- Let's look for singularities, what about $r = 2GM$? Check invariants $R = 0$ so it tells us nothing about singularities. Also $R_{\mu\nu} = 0$.
-
$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48G^2M^2}{r^6}$$

  So it does not look like $r = r_g = 2GM$ is a singularity.
- Radial light geodesics

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} = 0, \rightarrow \frac{dt}{dr} = \pm \frac{1}{1 - \frac{r_g}{r}}$$



Schwarzschild solution: singularity?

Although Eddington (1924) was the first to construct a coordinate system that is nonsingular at $r = 2M$, he seems not to have recognized the significance of his result. Lemaître (1933c, especially p. 82) appears to have been the first to recognize that the so-called "Schwarzschild singularity" at $r = 2M$ is not a singularity. He wrote, "La singularité du champ de Schwarzschild est donc une singularité fictive,

MTW book

Schwarzschild solution: Good coordinates

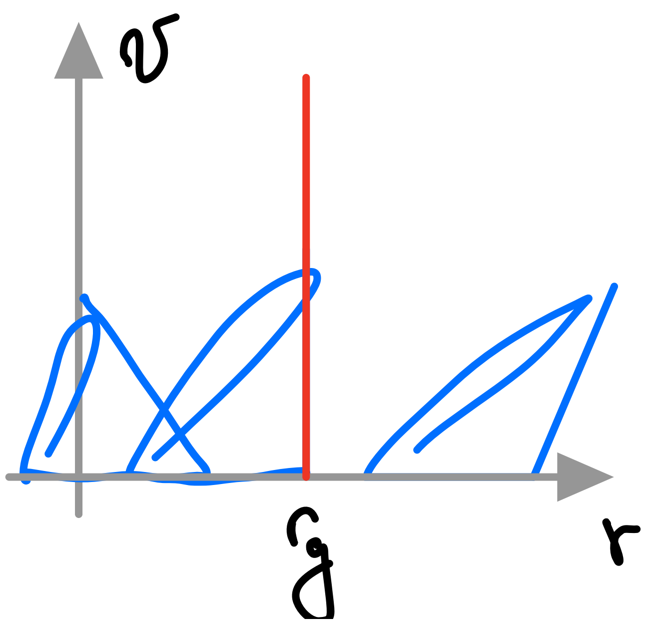
- Eddington-Finkelstein coordinates

$$r_* = r + r_g \ln\left|\frac{r}{r_g} - 1\right|, \quad v = t + r_*$$

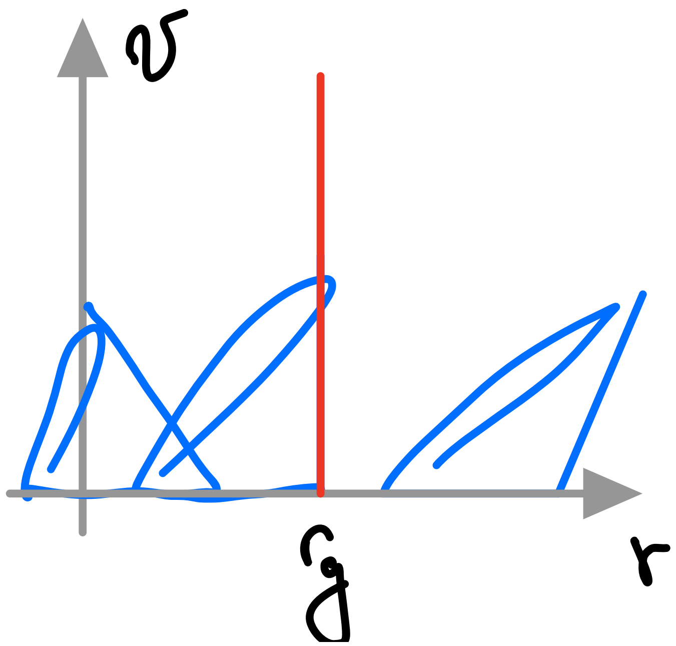
$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2$$

- Radial light geodesics

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dv^2 + 2dvdr = 0 \quad \rightarrow dv = 0, \frac{dv}{dr} = \pm\frac{2}{1 - \frac{r_g}{r}}$$



Schwarzschild solution: Horizon



- No curvature singularity at $r = r_g$ (in Schwarzschild coordinates it is a coordinate singularity)
- $r = r_g$ is a null surface.
- The Killing vector ∂_t becomes null at this hypersurface.
- ∂_t is null generator of the horizon.
- $r = r_g$ is a Killing horizon and also the event horizon.

The black hole is a region of space-time that is invisible to an outside or asymptotic observer. Loosely speaking an event horizon is then the boundary of this black hole region.

A null hypersurface is a Killing horizon $\mathcal{K}$ of a Killing vector field ξ if $\xi|_{\mathcal{K}}$ is normal to $\mathcal{K}$

Carter-Penrose (conformal) diagram

Main idea: to understand the causal structure of spacetime, we use conformal transformation

$$ds^2 \to d\tilde{s}^2 = \Omega(x)^2 ds^2$$

- The causal nature of a vector field or curve is invariant under conformal rescalings
- In particular, conformal rescalings preserve the lightcones $ds^2 = 0$ and thus the causal structure of the space-time encoded in the structure and behaviour of lightcones.
- In general, timelike or spacelike geodesics are not mapped into each other. However, the paths that are traced out by null geodesics are mapped into each other under conformal rescalings.

Carter-Penrose diagram for Minkowski spacetime

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \qquad -\infty < t < +\infty \text{ and } 0 \le r < +\infty$$

Light coordinates $\quad u = t - r \quad , \quad v = t + r$

$$ds^2 = -du\,dv + ((v-u)^2/4)d\Omega^2 \qquad -\infty < u \le v < +\infty$$

Let's introduce new coordinates

$$u = \tan U \quad , \quad v = \tan V \qquad -\pi/2 < U \le V < +\pi/2$$

The metric becomes

$$ds^2 = \frac{1}{4\cos^2 U \cos^2 V}\left(-4dU\ dV + \sin^2(V-U)d\Omega^2\right)$$

We remove the prefactor and consider the metric

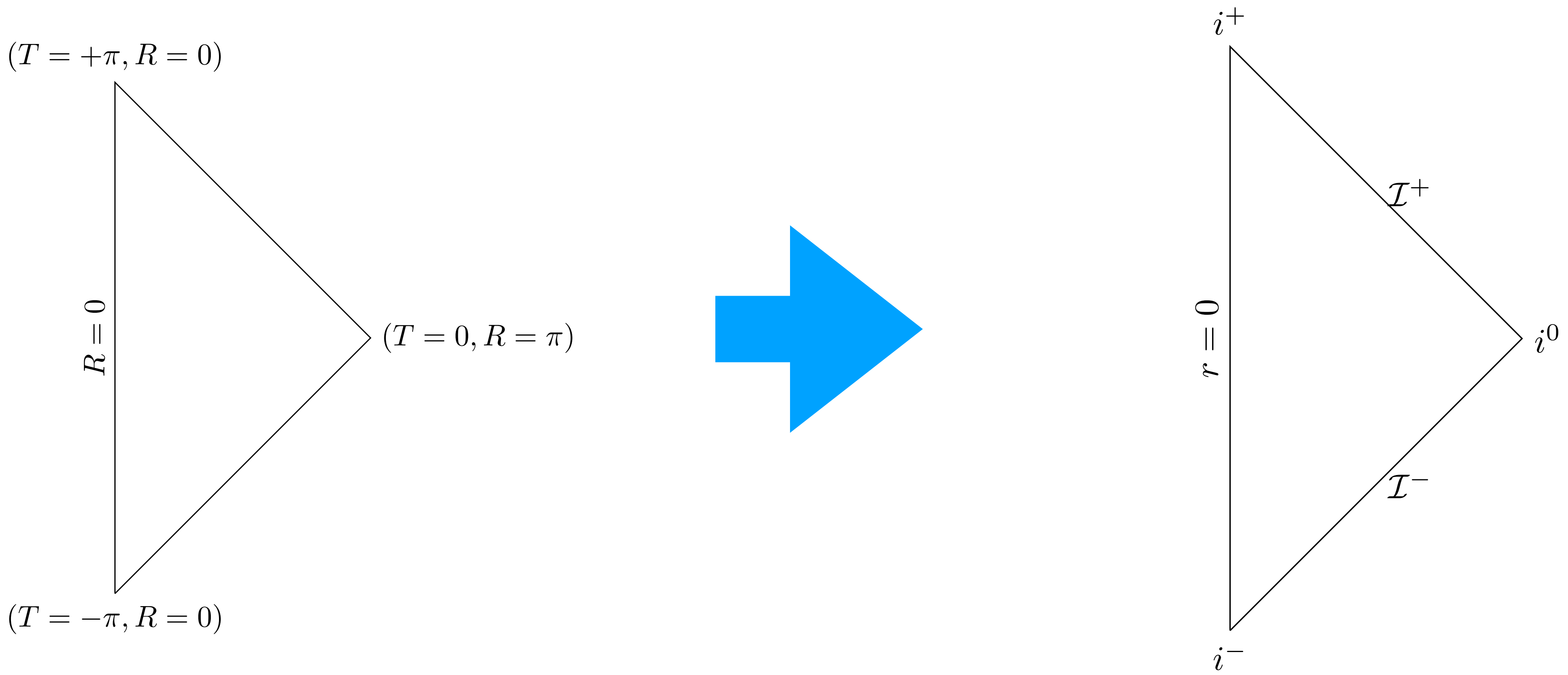
$$d\tilde{s}^2 = (4\cos^2 U \cos^2 V)ds^2 = -4dU\ dV + \sin^2(V-U)d\Omega^2$$

Carter-Penrose diagram for Minkowski spacetime

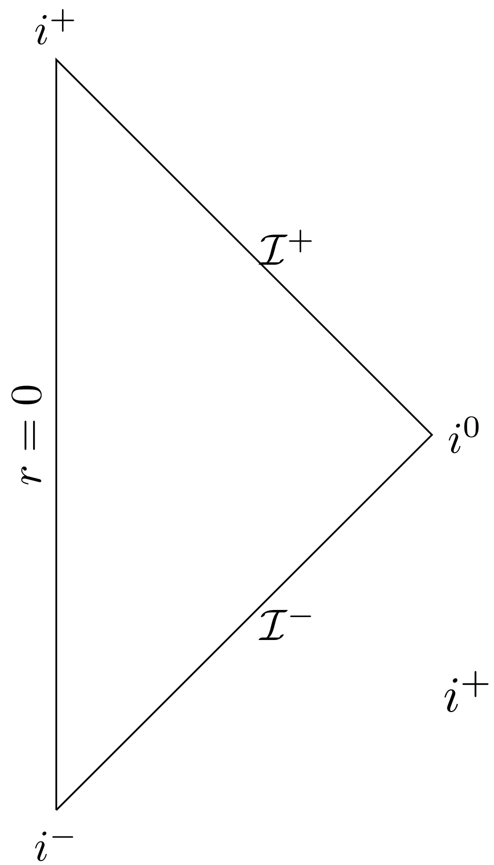
$$d\tilde{s}^2 = (4\cos^2 U \cos^2 V)ds^2 = -4dU\ dV + \sin^2(V-U)d\Omega^2$$

New coordinates (again) $\quad T = U + V \quad , \quad R = V - U \geq 0$

$$d\tilde{s}^2 = -dT^2 + dR^2 + \sin^2 R\ d\Omega^2 \qquad |T| + R < \pi \quad , \quad 0 \leq R < \pi$$



Carter-Penrose diagram for Minkowski spacetime



i^+ (*future timelike infinity*): where one asymptotes to when one takes $t \to +\infty$ at fixed r

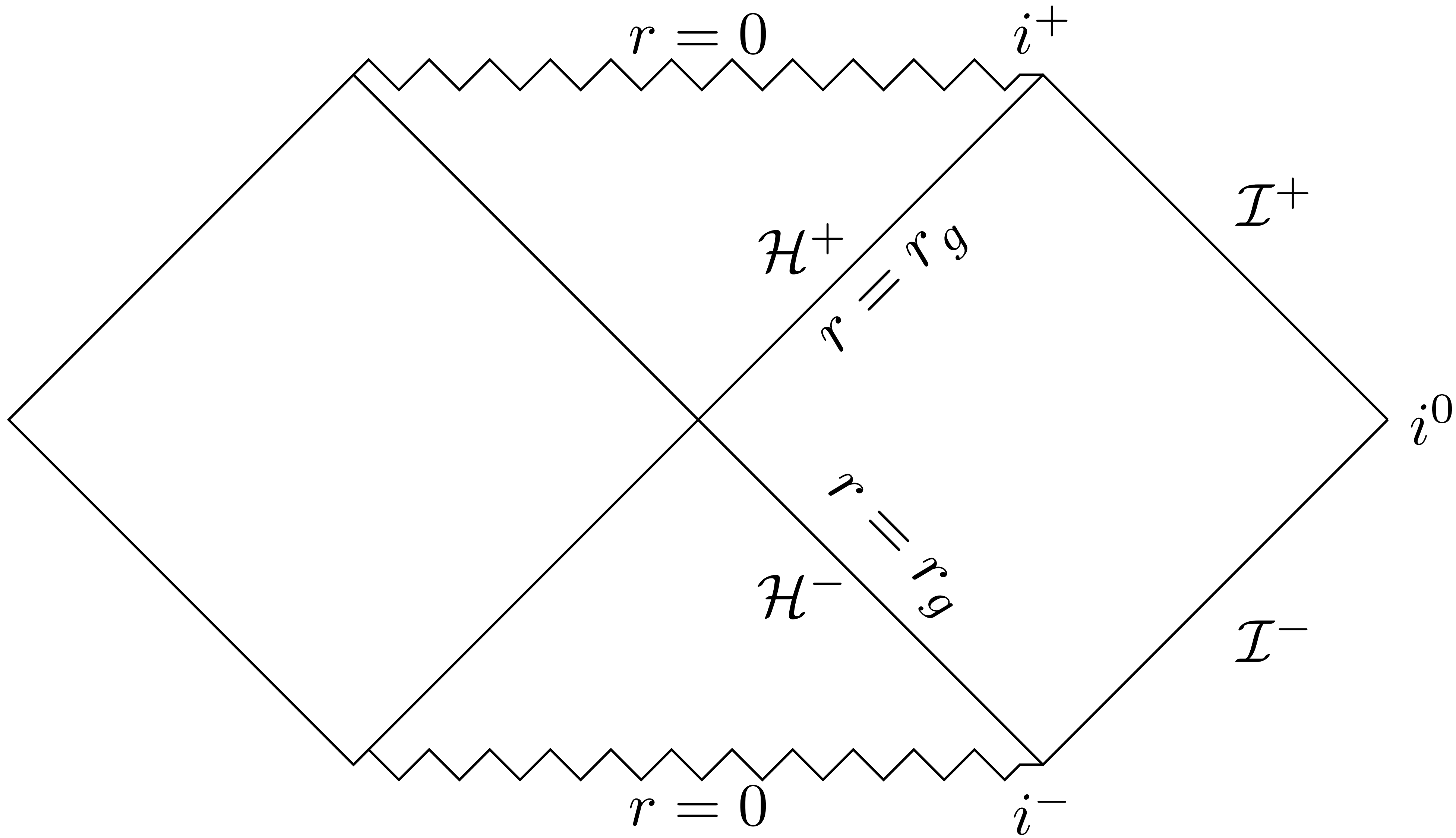
i^- (*past timelike infinity*): where one asymptotes to when one takes $t \to -\infty$ at fixed r

i^0 (*spacelike infinity*): where one asymptotes to when one instead takes $r \to \infty$ at fixed t

$\mathcal{I}^+$ (*future null infinity*): where outgoing radial lightrays asymptote to in the future, i.e. one takes $v \to \infty$ at fixed u

$\mathcal{I}^-$ (*past null infinity*): where ingoing radial lightrays asymptote to in the past, i.e. one takes $u \to -\infty$ at fixed v

Carter-Penrose diagram for (maximally extended) Schwarzschild



The future horizon $\mathcal{H}^+$ is the boundary of the region from which signals can escape to future null infinity $\mathcal{I}^+$

The horizon $\mathcal{H}^+$ is the boundary of (the closure of) the past of future null infinity.

Carter-Penrose diagram for a collapsing star