Neutrino physics (2)

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Neutrino oscillations

Neutrinos can oscillate !

A periodic change of neutrino flavour (identity):

$$\nu_e \to \nu_\mu \to \nu_e \to \nu_\mu \to \nu_e \quad \ldots$$

Happens without any external influence!

Dr. Jekyll / Mr. Hyde kind of story

Neutrinos have two-sided (or even 3-sided) personality !

$$\boxed{P(\nu_e \to \nu_\mu; L) = \sin^2 2\theta \cdot \sin^2\left(\frac{\Delta m^2}{4p} L\right)}$$

Hints of oscillations of solar neutrinos seen since the 1960s

First unambiguous evidence – oscillations of atmospheric neutrinos (The Super-Kamiokande Collaboration, 1998)

A bit of history...

Idea of neutrino oscillations: First put forward by Pontecorvo in 1957. Suggested possibility of $\nu \leftrightarrow \bar{\nu}$ oscillations by analogy with $K^0 \bar{K}^0$ oscillations.

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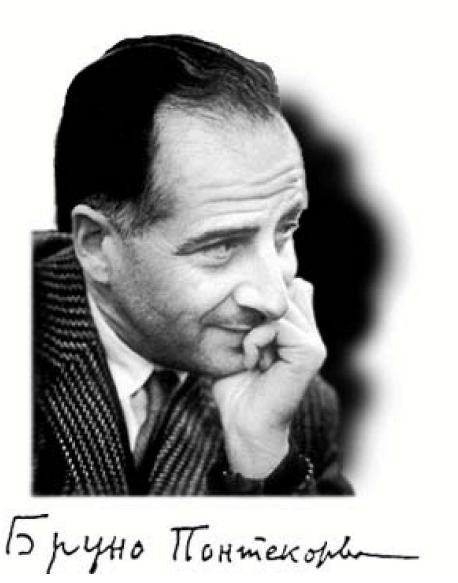
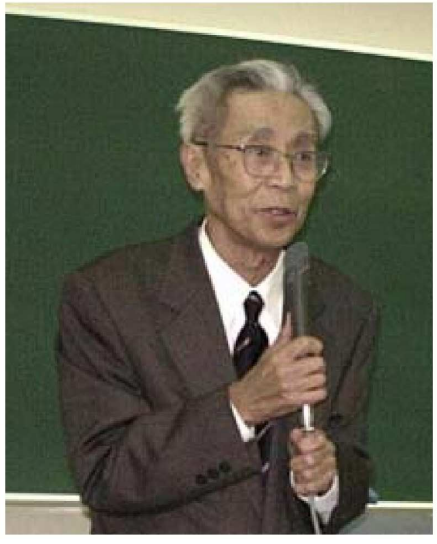
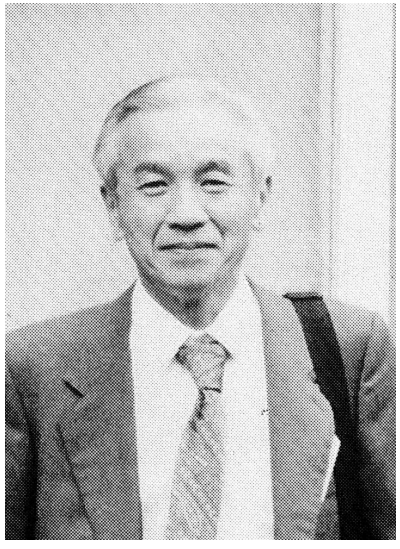


B. Pontecorvo
1913 - 1993

S. Sakata
1911 – 1970

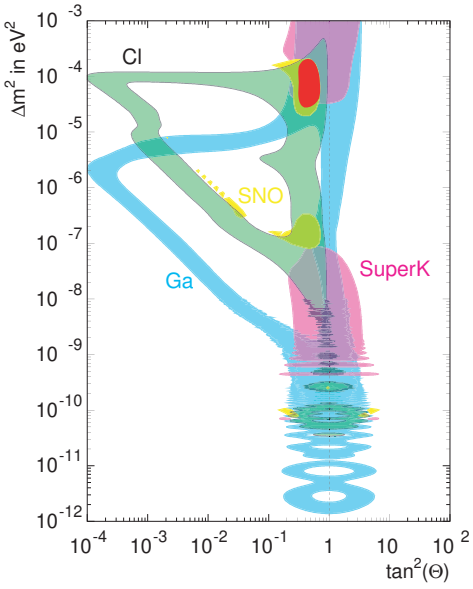


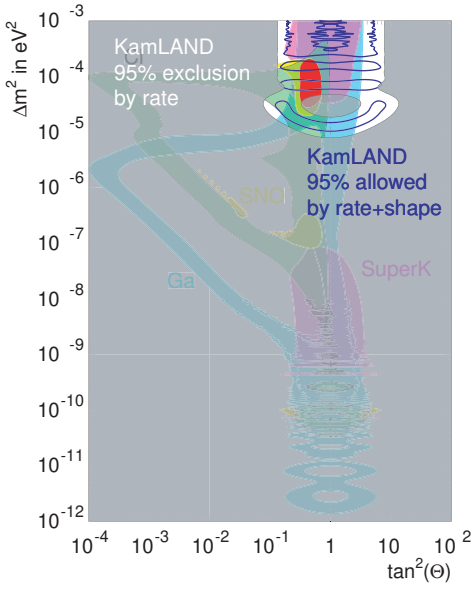
Z. Maki
1929 – 2005



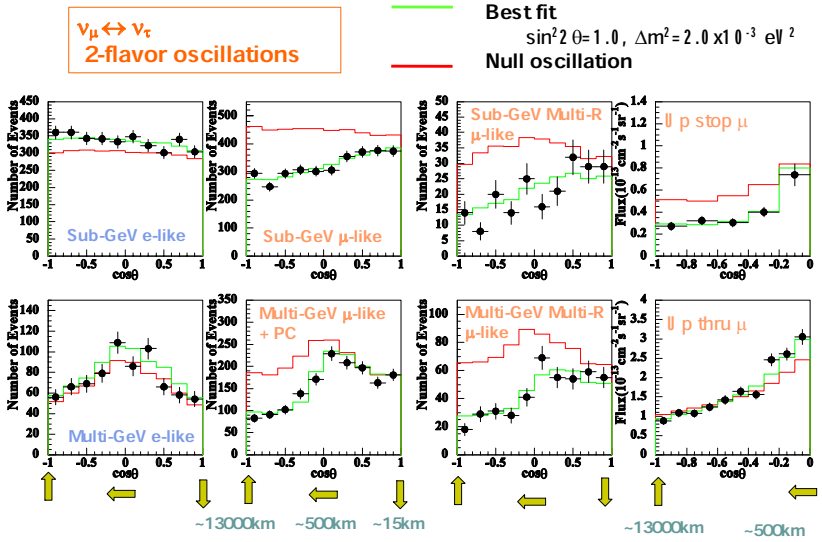
M. Nakagawa
1932 – 2001

Oscillations discovered experimentally !

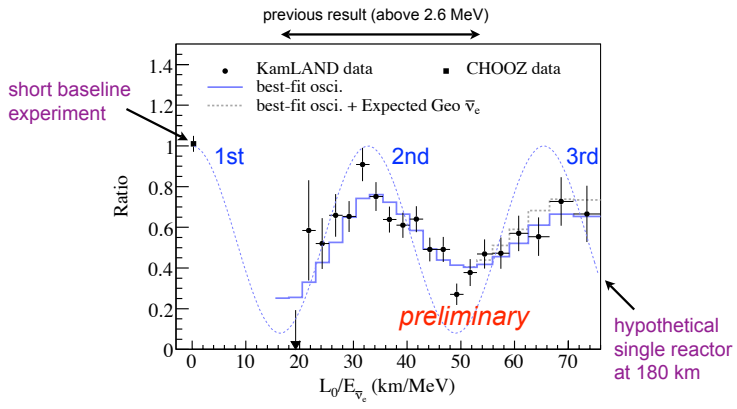




Zenith angle distributions



Neutrino Oscillation

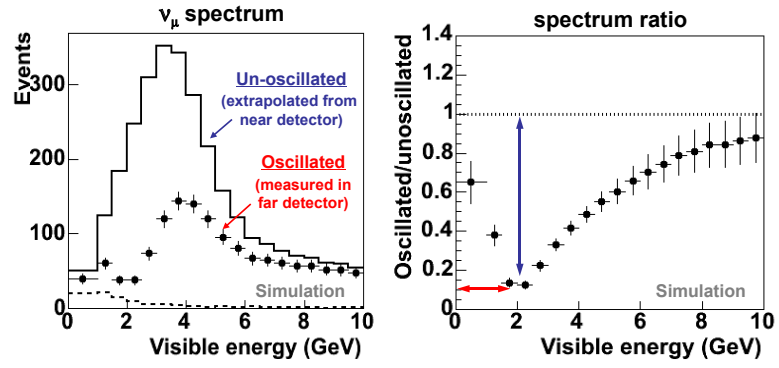


KamLAND covers the 2nd and 3rd maximum

→ **characteristic of neutrino oscillation**

ν_μ Disappearance Measurement

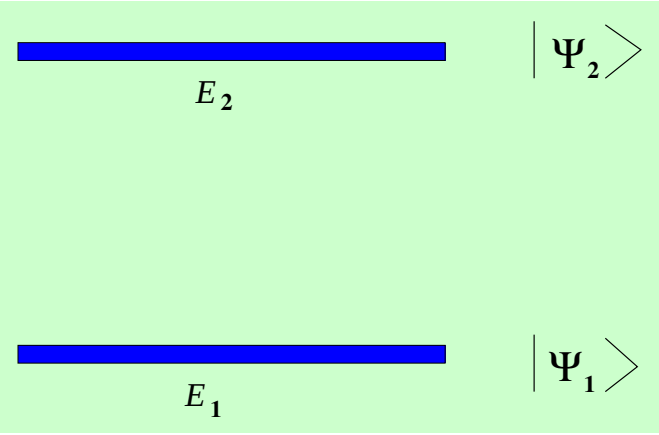
Look for ν_μ deficit : $P(\nu_\mu \to \nu_\mu) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{E}\right)$



Andy Blake, Cambridge University

The MINOS Experiment, slide 7

Oscillations: a well known QM phenomenon



$$\Psi_1(t) = e^{-i\,E_1\,t}\,\Psi_1(0)$$

$$\Psi_2(t) = e^{-i\,E_2\,t}\,\Psi_2(0)$$

$$\Psi(0) = a\,\Psi_1(0) + b\,\Psi_2(0) \quad (|a|^2 + |b|^2 = 1)\ ; \qquad \Rightarrow$$

$$\Psi(t) = a\,e^{-i\,E_1\,t}\,\Psi_1(0) + b\,e^{-i\,E_2\,t}\,\Psi_2(0)$$

Probability to remain in the same state $|\Psi(0)\rangle$ after time t:

$$\diamondsuit \quad P_{\rm surv} = |\langle\Psi(0)|\Psi(t)\rangle|^2 = \left||a|^2\,e^{-i\,E_1\,t} + |b|^2\,e^{-i\,E_2\,t}\right|^2$$

$$= 1 - 4|a|^2|b|^2\sin^2[(E_2 - E_1)\,t/2]$$

Neutrino oscillations: theory

Leptonic mixing

For $m_\nu \neq 0$ weak eigenstate neutrinos ν_e, ν_μ, ν_τ do not coincide with mass eigenstate neutrinos ν_1, ν_2, ν_3

Diagonalization of leptonic mass matrices:

$$e'_L \to V_L\, e_L\,, \qquad \nu'_L \to U_L\, \nu_L\, \ldots \qquad \Rightarrow$$

$$-\mathcal{L}_{w+m} = \frac{g}{\sqrt{2}} (\bar{e}_L \gamma^\mu V_L^\dagger U_L\, \nu_L)\, W_\mu^- + \text{diag. mass terms} + h.c.$$

Leptonic mixing matrix: $U = V_L^\dagger U_L$

$$\diamondsuit \qquad \nu_{\alpha L} = \sum_i U_{\alpha i}\, \nu_{iL} \qquad \Rightarrow \qquad |\nu_{\alpha L}\rangle = \sum_i U^*_{\alpha i}\, |\nu_{iL}\rangle$$

$$(\alpha = e\,,\, \mu\,,\, \tau, \qquad i = 1\,,\, 2\,,\, 3)$$

Master formula for ν oscillations

The standard formula for the oscillation probability of relativistic or quasi-degenerate in mass neutrinos in vacuum:

$$\diamondsuit \qquad \boxed{P(\nu_\alpha \to \nu_\beta; L) = \left| \sum_i U_{\beta i}\, e^{-i\frac{\Delta m^2_{ij}}{2p}L}\, U^*_{\alpha i} \right|^2}$$

$$(\hbar = c = 1)$$

Problem: prove that the RHS does not depend on the index j.

Oscillation disappear when either

- $U = \mathbb{1}$, i.e. $U_{\alpha i} = \delta_{\alpha i}$ (no mixing) or
- $\Delta m^2_{ij} = 0$ (massless or mass-degenerate neutrinos).

How is it usually derived?

Assume at time $t = 0$ and coordinate $x = 0$ a flavour eigenstate $|\nu_\alpha\rangle$ is produced:

$$|\nu(0,0)\rangle \;=\; |\nu_\alpha^{\rm fl}\rangle \;=\; \sum_i U^*_{\alpha i}\,|\nu_i^{\rm mass}\rangle$$

After time t at the position x, for <u>plane-wave particles</u>:

$$|\nu(t,\vec{x})\rangle \;=\; \sum_i U^*_{\alpha i}\,e^{-ip_i x}|\nu_i^{\rm mass}\rangle$$

Mass eigenstates pick up the phase factors $e^{-i\phi_i}$ with

$$\phi_i \;\equiv\; p_i\,x \;=\; Et \;-\; \vec{p}\,\vec{x}$$

$$P(\nu_\alpha \to \nu_\beta) \;=\; \left|\langle\nu_\beta^{\rm fl}|\nu(t,x)\rangle\right|^2$$

How is it usually derived?

Consider $\vec{x} \,||\, \vec{p} \quad \Rightarrow \quad \vec{p}\vec{x} = \mathrm{px} \quad (\mathrm{p} \;=\; |\vec{p}|, \quad \mathrm{x} \;=\; |\vec{x}|)$

Phase differences between different mass eigenstates:

$$\Delta\phi \;=\; \Delta E \cdot t \;-\; \Delta\mathrm{p} \cdot \mathrm{x}$$

Shortcuts to the standard formula

1. Assume the emitted neutrino state has a well defined momentum (same momentum prescription) $\Rightarrow \quad \Delta\mathrm{p} \;=\; 0.$

   For ultra-relativistic neutrinos $E_i \;=\; \sqrt{\mathrm{p}^2 + m_i^2} \;\simeq p + \frac{m_i^2}{2\mathrm{p}} \quad \Rightarrow$

$$\Delta E \;\simeq\; \frac{m_2^2 - m_1^2}{2E} \;\equiv\; \frac{\Delta m^2}{2E}\,; \qquad t \approx x \qquad (\hbar = c = 1)$$

$\Rightarrow$ The standard formula is obtained

How is it usually derived?

2. Assume the emitted neutrino state has a well defined energy (same energy prescription) $\Rightarrow \quad \Delta E = 0.$

$$\Delta\phi = \Delta E \cdot t - \Delta \mathrm{p} \cdot \mathrm{x} \quad \Rightarrow \quad - \Delta \mathrm{p} \cdot \mathrm{x}$$

For ultra-relativistic neutrinos $\mathrm{p}_i = \sqrt{E^2 - m_i^2} \simeq E - \frac{m_i^2}{2\mathrm{p}} \quad \Rightarrow$

$$-\Delta \mathrm{p} \equiv \mathrm{p}_1 - \mathrm{p}_2 \approx \frac{\Delta m^2}{2E};$$

$\Rightarrow$ The standard formula is obtained

Stand. phase $\Rightarrow$ $\boxed{(l_{\mathrm{osc}})_{ik} = \frac{4\pi E}{\Delta m_{ik}^2} \simeq 2.5\, m\, \frac{E\,(\mathrm{MeV})}{\Delta m_{ik}^2\, \mathrm{eV}^2}}$

Same E and same p approaches

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Very simple and transparent

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Allow one to quickly arrive at the desired result

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Trouble: they are both wrong

Kinematic constraints

Same momentum and same energy assumptions: contradict kinematics!

Pion decay at rest $(\pi^+ \to \mu^+ + \nu_\mu, \quad \pi^- \to \mu^- + \bar{\nu}_\mu)$:
For decay with emission of a massive neutrino of mass m_i:

$$E_i^2 = \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 + \frac{m_i^2}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_i^4}{4m_\pi^2}$$

$$p_i^2 = \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 - \frac{m_i^2}{2}\left(1 + \frac{m_\mu^2}{m_\pi^2}\right) + \frac{m_i^4}{4m_\pi^2}$$

For massless neutrinos: $E_i = p_i = E \equiv \frac{m_\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \simeq 30 \ \text{MeV}$

To first order in m_i^2:

$$E_i \simeq E + \xi\frac{m_i^2}{2E}, \qquad p_i \simeq E - (1-\xi)\frac{m_i^2}{2E}, \qquad \xi = \frac{1}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \approx 0.2$$

Kinematic constraints

Same momentum or same energy would require

$\xi = 1$ or $\xi = 0$ – not the case!

Also: would violate Lorentz invariance of the oscillation probability

How can wrong assumptions lead to the correct oscillation formula ?

Problems with the plane-wave approach

- Same momentum $\Rightarrow$ oscillation probabilities depend only on time. Leads to a paradoxical result – no need for a far detector ! "Time-to-space conversion" (??) – assumes neutrinos to be point-like particles (notion opposite to plane waves).
- Same energy – oscillation probabilities depend only on coordinate. Does not explain how neutrinos are produced and detected at certain times. Correspponds to a stationary situation.

Plane wave approach $\Leftrightarrow$ exact energy-momentum conservation. Neutrino energy and momentum are fully determined by those of external particles $\Rightarrow$ only one mass eigenstate can be emitted!

- Consistent approaches:

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- QM wave packet approach – neutrinos described by wave packets rather than by plane waves

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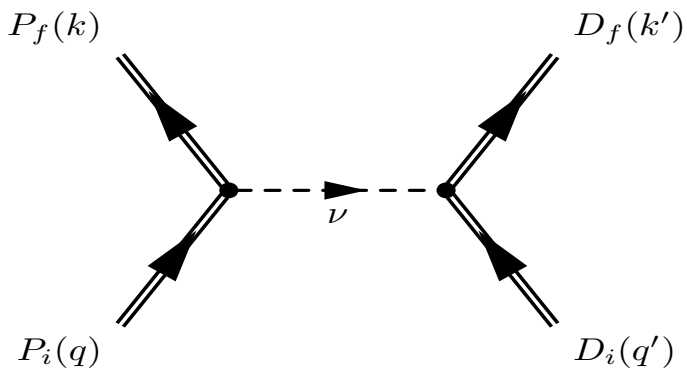
- QM wave packet approach – neutrinos described by wave packets rather than by plane waves
- QFT approach: neutrino production and detection explicitly taken into account. Neutrinos are intermediate particles described by propagators

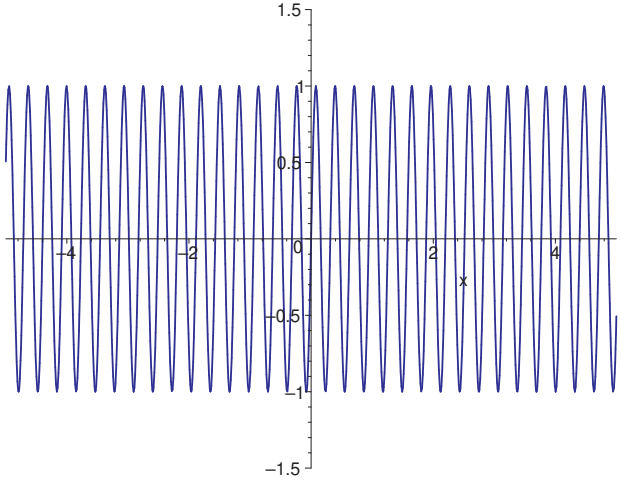


QM wave packet approach

In QM propagating particles are described by wave packets!

– Finite extensions in space and time.

Plane waves: the wave function at time $t = 0$ $\quad \Psi_{\vec{p}_0}(\vec{x}) \;=\; e^{i\vec{p}_0\vec{x}}$



Wave packets

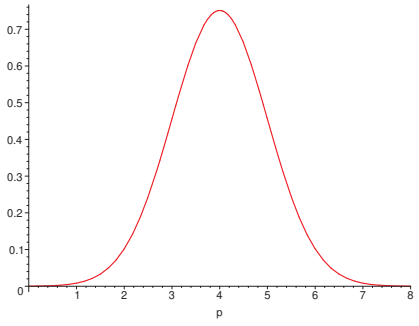
Wave packets: superpositions of plane waves with momenta in an interval of width σ_p around mom. p_0

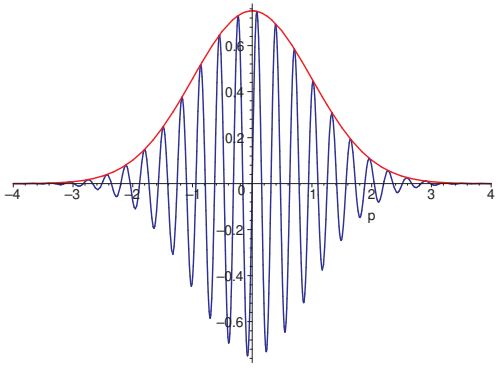
$$\sigma_x \, \sigma_p \geq 1/2 \quad - \quad \text{QM uncertainty relation}$$

W. packet centered at $\vec{x}_0 = 0$ at time $t = 0$:

$$\Psi(\vec{x};\, \vec{p}_0, \sigma_{\vec{p}}) \;=\; \int \frac{d^3p}{(2\pi)^3} \, f(\vec{p} - \vec{p}_0) \, e^{i\vec{p}\,\vec{x}}$$

Gaussian mom. space w. packet:





$$\sigma_x \sigma_p \;=\; 1/2 \quad - \quad \text{minimum uncertainty packet}$$

Propagating wave packets

Include time dependence:

$$\Psi(\vec{x},\, t) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f(\vec{p}-\vec{p}_0)\, e^{i\vec{p}\vec{x}-iE(p)t}$$

Example: Gaussian wave packets

Momentum-space distribution:

$$f(\vec{p}-\vec{p}_0) \;=\; \frac{1}{(2\pi\sigma_p^2)^{3/4}} \exp\left\{-\frac{(\vec{p}-\vec{p}_0)^2}{4\sigma_p^2}\right\}$$

Coordinate-space wave packet for ν_i (neglecting spreading):

$$\diamondsuit \quad \boxed{\Psi_i(\vec{x},\, t) \;=\; e^{i\vec{p}_0\vec{x}-iE_i(p_0)t}\; \frac{1}{(2\pi\sigma_x^2)^{3/4}} \exp\left\{-\frac{(\vec{x}-\vec{v}_{gi}t)^2}{4\sigma_x^2}\right\}}, \qquad \sigma_x^2 = 1/(4\sigma_p^2)$$

QM wave packet approach

The evolved produced state:

$$|\nu_\alpha^{\rm fl}(\vec{x},t)\rangle \;=\; \sum_i U_{\alpha i}^* \,|\nu_i^{\rm mass}(\vec{x},t)\rangle \;=\; \sum_i U_{\alpha i}^* \,\Psi_i^P(\vec{x},t)|\nu_i^{\rm mass}\rangle$$

Transition amplitude:

$$\mathcal{A}_{\alpha\beta}(T,\vec{L}) \;=\; \langle\nu_\beta^{\rm fl}|\nu_\alpha^{\rm fl}(T,\vec{L})\rangle \;=\; \sum_i U_{\alpha i}^* U_{\beta i}\,\mathcal{A}_i(T,\vec{L})$$

Strongly suppressed unless $|\vec{L}-\vec{v}_{gi}T| \lesssim \sigma_x$. E.g., for Gaussian wave packets:

$$\mathcal{A}_i(T,\vec{L}) \;\propto\; \exp\left[-\frac{(\vec{L}-\vec{v}_{gi}T)^2}{4\sigma_x^2}\right], \qquad \sigma_x^2 \;\equiv\; \sigma_{xP}^2+\sigma_{xD}^2$$

Phase difference

Oscillations are due to phase differences of different mass eigenstates:

$$\Delta\phi \;=\Delta E\cdot T \;-\; \Delta p\cdot L \qquad\qquad (E_i \;=\; \sqrt{p_i^2+m_i^2})$$

For relativistic or quasi-degenerate neutrinos: $\Delta E \ll E,\;\; \Delta p \ll p \quad \Rightarrow$

$$\Delta E \;=\; \frac{\partial E}{\partial p}\Delta p + \frac{\partial E}{\partial m^2}\Delta m^2 \;=\; v_g\,\Delta p \;+\; \frac{1}{2E}\,\Delta m^2$$

$$\Delta\phi \;=\; (v_g\,\Delta p + \frac{1}{2E}\Delta m^2)\;T \;-\; \Delta p\cdot L$$

$$= \;-\;(L \;-\; v_g\,T)\Delta p \;+\; \frac{\Delta m^2}{2E}\,T$$

In the center of wave packet $(L \;-\; v_g\,T) = 0$! In general, $|L \;-\; v_g\,T| \lesssim \sigma_x$;

if $\sigma_x\Delta p \ll 1$, $(\Delta p \ll \sigma_p,\; \sigma_x \ll l_{\rm osc}\,)$ $\quad\Rightarrow\quad$ $|L \;-\; v_g\,T|\Delta p \ll 1 \quad\Rightarrow$

$$\Delta\phi \;=\; \frac{\Delta m^2}{2E}\,T\,, \qquad\qquad L \;\simeq\; v_g T \;\simeq T$$

– the result of the “same momentum” approach recovered!

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- Neutrinos are relativistic or quasi-degenerate with $\Delta E \ll E$, $\Delta p \ll p$

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The reasons why wrong assumptions give the correct result:

- Neutrinos are relativistic or quasi-degenerate with $\Delta E \ll E$, $\Delta p \ll p$
- The size of the neutrino wave packet is small compared to the oscillation length: $\sigma_x \ll l_{\rm osc}$ (more precisely: energy uncertainty $\sigma_E \gg \Delta E$)

When are neutrino oscillations observable?

Keyword: *Coherence*

Neutrino flavour eigenstates ν_e, ν_μ and ν_τ are coherent superpositions of mass eigenstates ν_1, ν_2 and ν_3 $\Rightarrow$ oscillations are only observable if

- neutrino production and detection are coherent
- coherence is not (irreversibly) lost during neutrino propagation.

Possible decoherence at production (detection): If by accurate E and p measurements one can tell (through $E = \sqrt{p^2 + m^2}$) which mass eigenstate is emitted, the coherence is lost and oscillations disappear!

Full analogy with electron interference in double slit experiments: if one can establish which slit the detected electron has passed through, the interference fringes are washed out.

$\diamondsuit$ Decoherence is equivalent to averaging neutrino oscillations out.

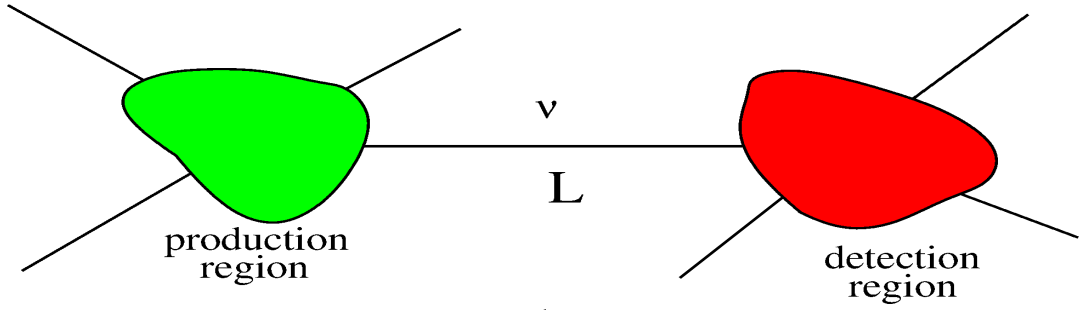
Oscillations: coherence of different ν_i

Usual assumption: the produced and detected neutrinos are flavour eigenstates

$$\diamondsuit \qquad |\nu_{\alpha L}\rangle = \sum_i U^*_{\alpha i} |\nu_{iL}\rangle \qquad (\alpha = e, \mu, \tau, \; i = 1, 2, 3)$$

$$\Updownarrow$$

oscillations



Intrinsic QM neutrino energy and momentum uncertainties (σ_E and σ_p) related to space-time localization of the production and detection processes play a crucial role.

Coherence vs. decoherence at ν production

E and p differences of neutrino mass eigenstates composing a flavour state:

$$\Delta E \equiv \Delta E_{ik} = \sqrt{p_i^2 + m_i^2} - \sqrt{p_k^2 + m_k^2}, \qquad \Delta p = p_i - p_k \,.$$

Production coherence condition (barring some cancellations): neutrino energy and momentum uncertainties must be sufficiently large to accommodate differing E_i and p_i:

$$\Delta E \ll \sigma_E, \qquad \Delta p \ll \sigma_p \,.$$

How are the oscillations destroyed when σ_E and σ_p are too small? Small σ_p means large uncertainty of the coordinate of neutrino production point. When it becomes larger than l_{osc} oscillations get washed out (Kayser 1981).

Configuration - space picture

Oscillation phase acquired over the distance x and time t:

$$\phi_{osc} = \Delta E \cdot t - \Delta p \cdot x \,.$$

Fluctuation of ϕ_{osc} due to uncertainty in 4-coordinate of neutrino production:

$$\delta\phi_{osc} = \Delta E \cdot \delta t - \Delta p \cdot \delta x \,,$$

δt and δx limited by the duration of the neutrino production process σ_t and its spatial extension σ_X: $\delta t \lesssim \sigma_t$, $|\delta x| \lesssim \sigma_X$.

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For oscillations to be observable $\delta\phi_{osc}$ must be small – otherwise oscillations will be washed out upon averaging over $(t_P,\, x_P)$ $\Rightarrow$

$$|\Delta E \cdot \delta t - \Delta p \cdot \delta x| \ll 1$$

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Barring accidental cancellations: $\Delta E \cdot \delta t \ll 1$, $\Delta p \cdot \delta x \ll 1$. From

$$\delta t \lesssim \sigma_t \sim \sigma_E^{-1}, \qquad \delta x \lesssim \sigma_X \sim \sigma_p^{-1} \qquad \Rightarrow$$

$$\diamondsuit \qquad \Delta E \ll \sigma_E, \qquad \Delta p \ll \sigma_p \,.$$

Different neutrino mass eigenstates are produced (detected) coherently and hence neutrino oscillations may be observable only if the oscillation phase acquired over the space-time extension of the production (detection) region is much smaller than unity.

Propagation decoherence

Another source of decoherence: wave packet separation due to the difference of group velocities Δv of different mass eigenstates.

If coherence is lost: Flavour transition can still occur, but in a non-oscillatory way. E.g. for $\pi \to \mu\nu_i$ decay with a subsequent detection of ν_i with the emission of e:

$$P \;\propto\; \sum_i P_{\rm prod}(\mu\,\nu_i) P_{\rm det}(e\,\nu_i) \;\propto\; \sum_i |U_{\mu i}|^2 |U_{ei}|^2$$

– the same result as for averaged oscillations.

Wave packet separation

Wave packets representing different mass eigenstate components have different group velocities v_{gi} $\Rightarrow$ after time $t_{\rm coh}$ (coherence time) they separate $\Rightarrow$ Neutrinos stop oscillating! (Only averaged effect observable).

Coherence time and length:

$$\Delta v \cdot t_{\rm coh} \simeq \sigma_x \,; \qquad l_{\rm coh} \simeq v t_{\rm coh}$$

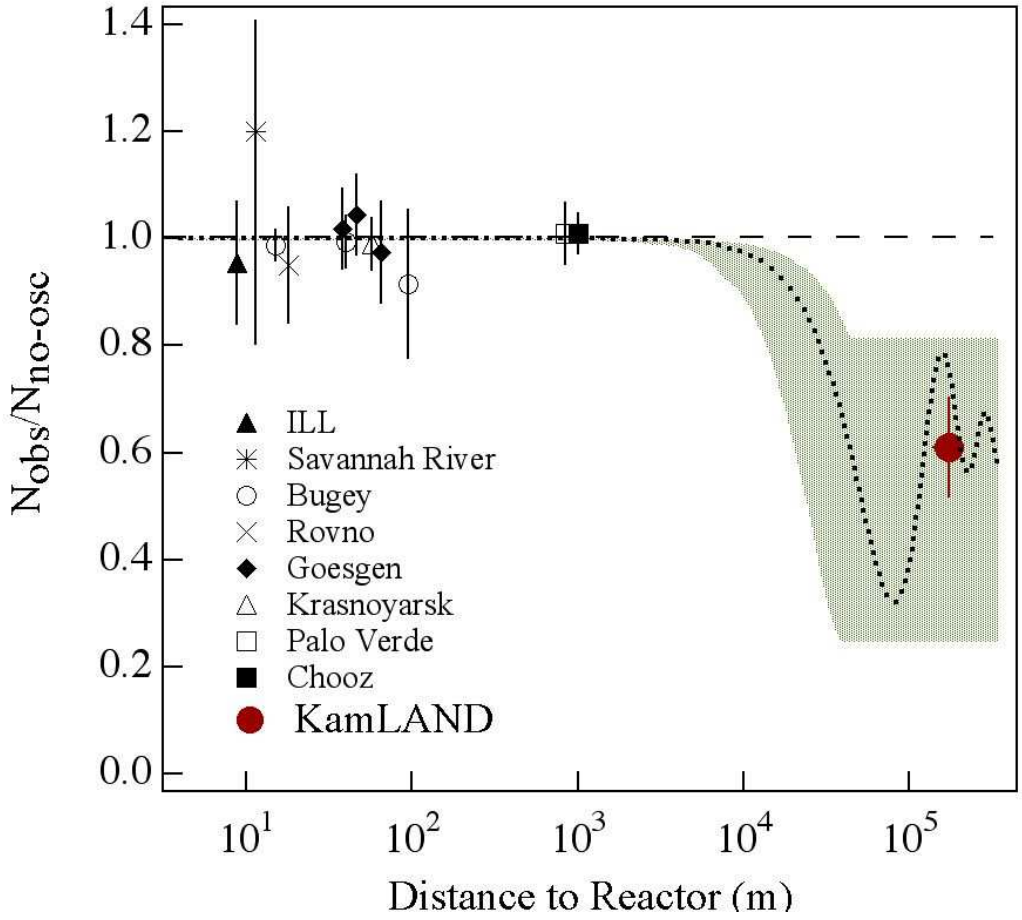
$$\Delta v = \frac{p_i}{E_i} - \frac{p_k}{E_k} \simeq \frac{\Delta m^2}{2E^2}$$

$$\boxed{l_{\rm coh} \simeq \frac{v}{\Delta v}\sigma_x = \frac{2E^2}{\Delta m^2}\, v\sigma_x}$$

The standard formula for $P_{\rm osc}$ is obtained when the decoherence effects are negligible.

A manifestation of neutrino coherence –

Non-observation of neutrino oscillations at short distances.



Expected: 365.2 ± 23.7

Background: 17.8 ± 7.3

Observed: 258

A manifestation of neutrino coherence

Even non-observation of neutrino oscillations at distances $L \ll l_{\rm osc}$ is a consequence of and an evidence for coherence of neutrino emission and detection! Two-flavour example (e.g. for ν_e emission and detection):

$$A_{\rm prod/det}(\nu_1) \sim \cos\theta\,, \qquad A_{\rm prod/det}(\nu_2) \sim \sin\theta \qquad \Rightarrow$$

$$A(\nu_e \to \nu_e) = \sum_{i=1,2} A_{\rm prod}(\nu_i) A_{\rm det}(\nu_i) = \cos^2\theta + e^{-i\Delta\phi}\sin^2\theta$$

Phase difference $\Delta\phi$ vanishes at short L $\Rightarrow$

$$P(\nu_e \to \nu_e) = (\cos^2\theta + \sin^2\theta)^2 = 1$$

If ν_1 and ν_2 were emitted and absorbed incoherently $\Rightarrow$ one would have to sum probabilities rather than amplitudes:

$$P(\nu_e \to \nu_e) \sim \sum_{i=1,2} |A_{\rm prod}(\nu_i)|^2 |A_{\rm det}(\nu_i)|^2 \sim \cos^4\theta + \sin^4\theta \; < \; 1$$

For Gaussian WPs:

Giunti, Kim & Lee, Phys. Lett. B274 (1992) 87:

$$\boxed{P_{\alpha\beta}(L,E) = \sum_{i,k} U_{\alpha i} U^*_{\beta i} U^*_{\alpha k} U_{\beta k} e^{-i(\Delta m^2_{ik}/2p)L}\, e^{-[L/(l_{\rm coh})_{ik}]^2 - [\Delta E^2_{ik}/8\sigma^2_E]}}$$

$$(l_{\rm coh})_{ik} = 2\sqrt{2}\frac{v_g}{|\Delta v_g|}\sigma_x = 2\sqrt{2}\frac{2E^2}{|\Delta m^2_{ik}|}\sigma_x\,; \qquad \sigma_x = 1/2\sigma_p = (1/2)(v_g/\sigma_E)$$

$$\frac{1}{\sigma^2_E} = \frac{1}{\sigma^2_{Eprod}} + \frac{1}{\sigma^2_{Edet}}$$

$$\Delta E_{ik} = \xi\frac{\Delta m^2_{ik}}{2E}$$

◇ Overall normalization obtained by imposing unitarity condition!

Are coherence constraints compatible?

Observability conditions for ν oscillations:

- Coherence of ν production and detection
- Coherence of ν propagation

Both conditions put upper limits on neutrino mass squared differences Δm^2 :

$$(1) \quad \Delta E_{jk} \sim \frac{\Delta m^2_{jk}}{2E} \ll \sigma_E; \qquad\qquad (2) \quad \frac{\Delta m^2_{jk}}{2E^2} L \ll \sigma_x \simeq v_g/\sigma_E$$

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But: The constraints on σ_E work in opposite directions:

$$(1) \quad \Delta E_{jk} \sim \frac{\Delta m^2_{jk}}{2E} \ll \sigma_E \ll \frac{2E^2}{\Delta m^2_{jk}} \frac{v_g}{L} \quad (2)$$

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Are they compatible? – Yes, if LHS $\ll$ RHS $\Rightarrow$

$$\boxed{2\pi \frac{L}{l_{\rm osc}} \ll \frac{v_g}{\Delta v_g} \; (\gg 1)}$$

– fulfilled in all cases of practical interest

Are coherence conditions satisfied?

The coherence propagation condition: satisfied very well for all but astrophysical and cosmological neutrinos (solar, SN, relic ν's ...)

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Sterile neutrinos: hints from SBL accelerator experiments (LSND, MiniBooNE), reactor neutrino anomaly, keV sterile neutrinos, pulsar kicks, leptogenesis via ν oscillations, SN r-process nucleosynthesis, unconventional contributions to $2\beta0\nu$ decay ...

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Sterile neutrinos: hints from SBL accelerator experiments (LSND, MiniBooNE), reactor neutrino anomaly, keV sterile neutrinos, pulsar kicks, leptogenesis via ν oscillations, SN r-process nucleosynthesis, unconventional contributions to $2\beta0\nu$ decay ...

Production/detection coherence has to be re-checked – important implications for some neutrino experiments!

Neutrino oscillations: *Coherence at macroscopic distances –*

$L > 10{,}000$ km in atmospheric neutrino experiments !

Universal oscillation formula?

The complete process: production – propagation – detection: factorization

$$\Gamma_{ab} \;=\; j_a(E)\, P_{ab}^{\rm prop}(L,E)\, \sigma_b(E)$$

with a universal $P_{ab}^{\rm prop}(L,E)$ is only possible when all 3 processes are independent

In general not true, and production – propagation – detection should be considered as a single inseparable process!

To get the standard formula one assumes for the emitted and absorbed states

$$|\nu_a^{\rm fl}\rangle \;=\; \sum_i U_{ai}^*\, |\nu_i^{\rm mass}\rangle$$

The weights of the mass eigenstates are just U_{ai}^* – do not depend on the masses of ν_i $\Rightarrow$ only true when the phase space volumes at production and detection do not depend on the mass of ν_i.

Universal oscillation formula?

This is only true if the charact. energy E at production (and detection) is large compared to all m_i (relativistic neutrinos), or compared to all $|m_i - m_k|$ (quasi-degenerate neutrinos).

$\Rightarrow$ Neutrino oscillations can be described by a universal probability only when neutrinos are relativistic or quasi-degenerate

Also: degree of coherence of the propagating neutrino state depends on the coherence of the production and detection processes

$\Rightarrow$ The standard formula for the oscillation probability is only valid when all decoherence effects are negligible !

Oscillation probability in vacuum

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The standard formula for osc. probability is stubbornly robust.

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But: Conditions for partial decoherence are difficult to realize

They may still be realized if relatively heavy sterile neutrinos exist

Phenomenology of neutrino oscillations

An important example: 2-flavour case

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$
$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

$\Rightarrow$

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \equiv \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$

***

$$\diamondsuit \quad P_{\rm tr} = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2}{4p}L\right)$$

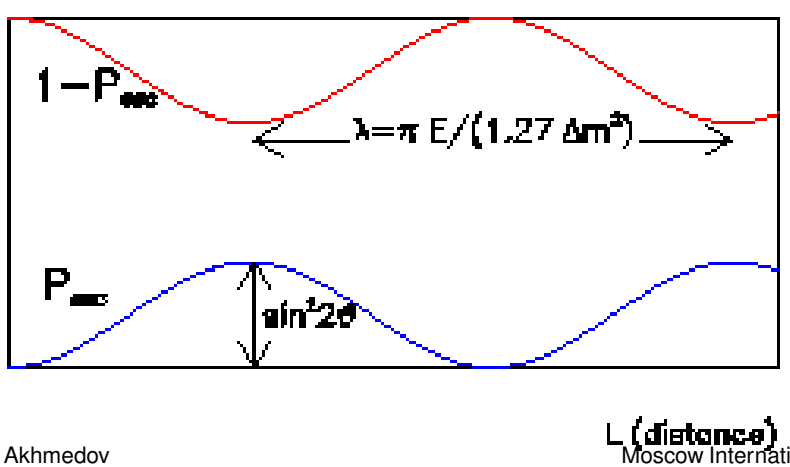
- Problem: Derive this formula from the general expression for $P_{\alpha\beta}$.
- Problem: Write this formula in the usual units, reinstating all factors of $\hbar$ and c. Find its classical and non-relativistic limits.

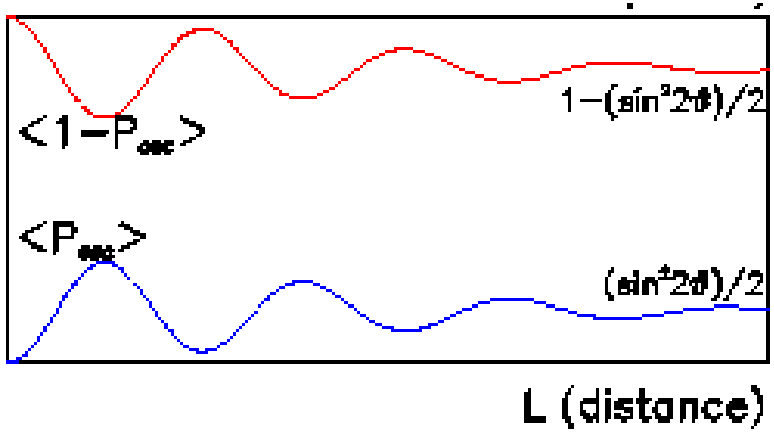
Oscillation amplitude: $\sin^2 2\theta$. Oscillation phase:

$$\frac{\Delta m^2}{4p} L = \pi \frac{L}{l_{\rm osc}}\,, \qquad l_{\rm osc} \equiv \frac{4\pi p}{\Delta m^2} \simeq 2.48\,{\rm m}\,\frac{p\,({\rm MeV})}{\Delta m^2\,({\rm eV}^2)}\,.$$

For large oscillation phase $\Rightarrow$ averaging regime (due to finite E-resolution of detectors and/or finite size of ν source/detector):

$$P_{\rm tr} = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2}{4p} L\right) \quad \to \quad \frac{1}{2}\sin^2 2\theta$$





2f evolution equation in vacuum

For relativistic point-like ν's $(x \simeq t)$ the evolution equation in the flavour basis:

$$i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = H_{\rm fl}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \left[U\begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}U^\dagger\right]\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}$$

$E \simeq p + \frac{m^2}{2E} \quad \Rightarrow$

$$H_{\rm fl} \simeq \left[U\begin{pmatrix}p+\frac{m_1^2}{2E} & 0\\ 0 & p+\frac{m_2^2}{2E}\end{pmatrix}U^\dagger\right] \Longrightarrow \left[U\begin{pmatrix}-\frac{\Delta m_{21}^2}{4E} & 0\\ 0 & \frac{\Delta m_{21}^2}{4E}\end{pmatrix}U^\dagger\right]$$

N.B.: A term prop. to unit matrix can always be added to/subtracted from $H_{\rm fl}$. Problem: prove this!

2-flavor evolution equation:

$$\diamondsuit \quad i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \begin{pmatrix}-\frac{\Delta m^2}{4E}\cos 2\theta & \frac{\Delta m^2}{4E}\sin 2\theta\\ \frac{\Delta m^2}{4E}\sin 2\theta & \frac{\Delta m^2}{4E}\cos 2\theta\end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}$$

- Problem: find P_{tr} by solving the evolution equation with the initial contition $(1,0)^T$.

Oscillation parameters as characteristics of H

For a 2×2 real symmetric matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

the angle of rotation that diagonalizes it:

$$\tan 2\theta \;=\; \frac{2b}{c-a}.$$

Eigenvalues:

$$\lambda_{1,2} \;=\; \frac{a+c}{2} \mp \sqrt{\frac{(c-a)^2}{4} + b^2}\,.$$

$$\longleftrightarrow$$

Mixing angle θ: the angle of rotation that diagonalizes eff. Hamiltonian $H_{\rm fl}$.

Eigenvalues of $H_{\rm fl}$: $\mathcal{E}_{1,2} = \pm\frac{\Delta m^2}{4E}$.

Oscillation length: $\boxed{l_{\rm osc} \;=\; \frac{2\pi}{|\mathcal{E}_2 - \mathcal{E}_1|} v_g \;=\; \frac{4\pi p}{\Delta m^2}}$

3f neutrino mixing and oscillations

General case of n flavours – parameter counting

$(n \times n)$ unitary mixing matrix $\tilde{U}$ $\Rightarrow$ n^2 real parameters:

$$\begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{n(n-1)}{2} \quad \text{mixing angles}, \qquad \frac{n(n+1)}{2} \quad \text{phases}$$

For leptonic mixing matrix n phases can be absorbed into re-defenition of the phases of LH charged fields: $e_{\alpha L} \to e^{i\phi_\alpha} e_{\alpha L}$ (e.g., 1st line of $\tilde{U}$ can be made real). This can be compensated in the mass term of charged leptons by rephasing $e_{\alpha R} \to e^{i\phi_\alpha} e_{\alpha R}$, so that $\bar{e}_{\alpha L} e_{\alpha R} = inv$.

Similarly, for <u>Dirac</u> neutrinos phases of one column can be fixed by absorbing $n-1$ phases into a redefinition of ν_{iL} (RH neutrino fields can be rephased analogously, so that $\bar{\nu}_{iL}\nu_{iR} = inv$.) $\Rightarrow$ In Dirac ν case $n + (n-1) = 2n - 1$ phases are unphysical – can be rotated away by redefining charged lepton and neutrino fields.

N.B.: Kinetic terms of e_L, e_R and ν_L, ν_R are also invariant w.r.t. rephasing.!

Physical phases

Number of physical phases:

$$\frac{n(n+1)}{2} - (2n-1) \;=\; \frac{(n-1)(n-2)}{2}.$$

Phys. phases responsible for CP violation! $\Rightarrow$ No Dirac-type CPV for $n < 3$.

In Majorana case:

$$\mathcal{L}_m \propto \nu_L^T C \nu_L \;+\; h.c.$$

Rephasing of ν_L is not possible (cannot be compensated in $\mathcal{L}_m$)

Only n phases can be removed from $\tilde{U}$ (by redefinition of $e_{\alpha L}$ fields) $\Rightarrow$
In addition to Dirac-type phases there are $(n-1)$ physical Majorana-type CP-violating phases.

Majorana phases do not affect oscillations

Majorana-type phases can be factored out in the mixing matrix:

$$\tilde{U} = U K$$

U contains Dirac-type phases, K – Majorana-type phases σ_i:

$$K = \mathrm{diag}(1\,, e^{i\sigma_1}\,, \ldots, e^{i\sigma_{n-1}})$$

Neutrino evolution equation: $i\frac{d}{dt}\,\nu = H_{\rm eff}\,\nu$

$$H_{\rm eff} = U K \begin{pmatrix} E_1 & & & \\ & E_2 & & \\ & & . & \\ & & & . \end{pmatrix} K^\dagger U^\dagger = U \begin{pmatrix} E_1 & & & \\ & E_2 & & \\ & & . & \\ & & & . \end{pmatrix} U^\dagger$$

Does not depend on the matrix of Majorana $\not{CP}$ phases K $\Rightarrow$
ν oscillations are insensitive to Majorana phases. Also true for osc. in matter.

3f oscillation parameters

Three neutrino species $(\nu_e, \nu_\mu, \nu_\tau)$ – linear superpositions of three mass eigenstates (ν_1, ν_2, ν_3). Mixing matrix U – 3×3 unitary matrix. Depends on 3 mixing angles and one Dirac-type $\not{CP}$ phase $\delta_{\rm CP}$.

Experiment: 2 mixing angles large (in the standard parameterization – θ_{12} and θ_{23}), one (θ_{13}) is relatively small.

Three neutrinos species – 2 independent mass squared differences, e.g. Δm^2_{21} and Δm^2_{31}.

$$\Delta m^2_{21} \ll \Delta m^2_{31}$$

What do we know about neutrino parameters

From atmsopheric and LBL accelerator neutrino experiments:

$$\diamondsuit \quad \Delta m^2_{31} \simeq 2.5 \times 10^{-3}\ \mathrm{eV}^2\,, \qquad \theta_{23} \sim 45^\circ$$

From solar neutrino experiments and KamLAND:

$$\diamondsuit \quad \Delta m^2_{21} \simeq 7.5 \times 10^{-5}\ \mathrm{eV}^2\,, \qquad \theta_{12} \simeq 33^\circ$$

From T2K + Double Chooz, Daya Bay and Reno reactor neutrino experiments:

$$\diamondsuit \quad \theta_{13} \simeq 9^\circ \quad (\text{previosly from Chooz} \lesssim 12^\circ)$$

CP-violating phase δ_{CP} practically unconstrained at the moment.

Leptonic mixing and 3f osc. in vacuum

Relation between flavour and mass eigenstates:

$$\nu_\alpha = \sum_{i=1}^{3} U_{\alpha i}\, \nu_i$$

ν_α – fields of flavour eigenstates, ν_i – of mass eigenstates.

3f mixing matrix:

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

Leptonic mixing and 3f osc. in vacuum

Relation btween flavour and mass eigenstates:

$$|\nu_\alpha\rangle = \sum_{i=1}^{3} U^*_{\alpha i}\,|\nu_i\rangle$$

Oscillation probability in vacuum:

$$P(\nu_\alpha \to \nu_\beta; L) \;=\; \left|\sum_{i=1}^{3} U_{\beta i}\; e^{-i\frac{\Delta m^2_{i1}}{2p}L}\; U^*_{\alpha i}\right|^2 \;=\; \left|\left[U\; e^{-i\frac{\Delta m^2}{2p}L}\; U^\dagger\right]_{\beta\alpha}\right|^2$$

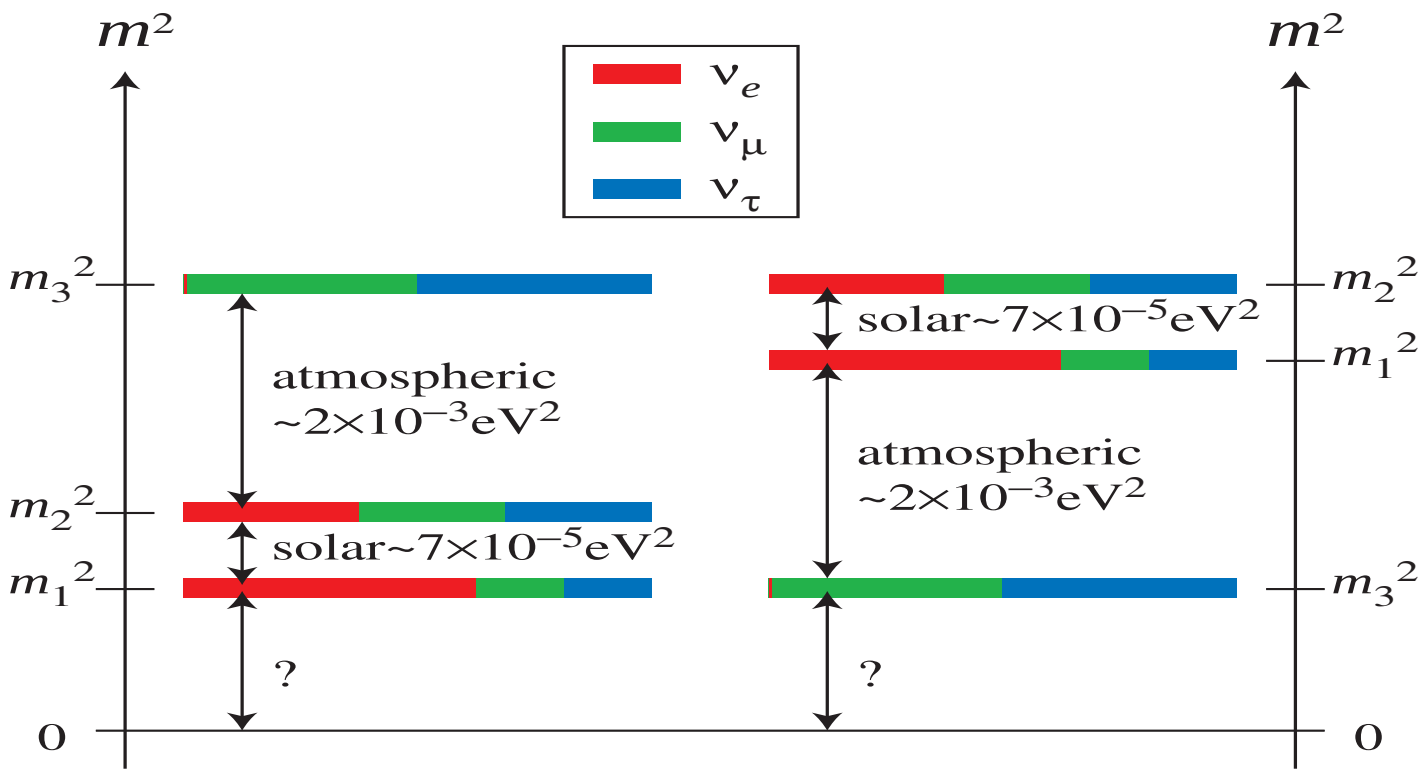
3f mixing matrix in the standard parameterization ($c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$):

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\rm CP}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{\rm CP}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= O_{23}\,(\Gamma_\delta\, O_{13}\, \Gamma^\dagger_\delta)\, O_{12}\,, \qquad \Gamma_\delta \equiv \mathrm{diag}(1\,,\; 1\,,\; e^{i\delta_{\rm CP}})$$

3f neutrino mixing

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\rm CP}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{\rm CP}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{\rm CP}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{\rm CP}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{\rm CP}} & c_{13}c_{23} \end{pmatrix}$$



$\not{CP}$ and $\not{T}$ in ν osc. in vacuum

$\nu_a \to \nu_b$ oscillation probability:

$$\diamondsuit \quad P(\nu_\alpha, t_0 \to \nu_\beta; t) = \left| \sum_i U_{\beta i}\, e^{-i\frac{\Delta m^2_{i1}}{2E}(t-t_0)}\, U^*_{\alpha i} \right|^2$$

- CP: $\nu_{\alpha,\beta} \leftrightarrow \bar{\nu}_{\alpha,\beta} \quad \Rightarrow \quad U_{\alpha i} \to U^*_{\alpha i} \quad (\{\delta_{\rm CP}\} \to -\{\delta_{\rm CP}\})$
- T: $t \rightleftarrows t_0 \quad \Leftrightarrow \quad \nu_\alpha \leftrightarrow \nu_\beta$

  $\Rightarrow \quad U_{\alpha i} \to U^*_{\alpha i} \quad (\{\delta_{\rm CP}\} \to -\{\delta_{\rm CP}\})$

T-reversed oscillations ("backwards in time") $\Leftrightarrow$ oscillations between interchanged initial and final flavours

- $\diamond$ *$\not{CP}$ and $\not{T}$ – absent in 2f case, pure $N \geq 3$f effects!*
- $\diamond$ No $\not{CP}$ and $\not{T}$ for survival probabilities ($\beta = \alpha$).

CP and T violation in vacuum – contd.

- CPT: $\nu_{\alpha,\beta} \leftrightarrow \bar{\nu}_{\alpha,\beta}$ & $t \rightleftarrows t_0$ $(\nu_\alpha \leftrightarrow \nu_\beta)$

  ◇ $P(\nu_\alpha \to \nu_\beta) \to P(\bar{\nu}_\beta \to \bar{\nu}_\alpha)$

The standard formula for $P_{\alpha\beta}$ in vacuum is CPT invariant!

$\not{CP} \Leftrightarrow \not{T}$ – *consequence of CPT*

Measures of $\not{CP}$ and $\not{T}$ – probability differences:

$$\Delta P^{\rm CP}_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)$$

$$\Delta P^{\rm T}_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta) - P(\nu_\beta \to \nu_\alpha)$$

From CPT:

◇ $\Delta P^{\rm CP}_{\alpha\beta} = \Delta P^{\rm T}_{\alpha\beta}\,;$ $\qquad \Delta P^{\rm CP}_{\alpha\alpha} = 0$

3f case

One $\not{CP}$ Dirac-type phase $\delta_{\rm CP}$ *(Majorana phases do not affect ν oscillations!)* $\Rightarrow$ one $\not{CP}$ and $\not{T}$ observable:

$$\diamond \quad \Delta P^{\rm CP}_{e\mu} = \Delta P^{\rm CP}_{\mu\tau} = \Delta P^{\rm CP}_{\tau e} \equiv \Delta P$$

$$\Delta P = -4 s_{12}\, c_{12}\, s_{13}\, c_{13}^2\, s_{23}\, c_{23}\, \sin\delta_{\rm CP} \times \left[\sin\left(\frac{\Delta m^2_{12}}{2E}\, L\right) + \sin\left(\frac{\Delta m^2_{23}}{2E}\, L\right) + \sin\left(\frac{\Delta m^2_{31}}{2E}\, L\right)\right]$$

Vanishes when

- At least one $\Delta m^2_{ij} = 0$
- At least one $\theta_{ij} = 0$ or 90°
- $\delta_{\rm CP} = 0$ or 180°
- In the averaging regime
- In the limit $L \to 0$ (as L^3)

Very difficult to observe!

Small parameters

Approximate formulas for probabilities can be obtained using expansions in small parameters:

$$(1) \quad \frac{\Delta m^2_{\odot}}{\Delta m^2_{\rm atm}} = \frac{\Delta m^2_{21}}{\Delta m^2_{31}} \sim 1/30$$

$$(2) \quad |U_{e3}| = |\sin\theta_{13}| \sim 0.16$$

In the limits $\Delta m^2_{21} = 0$ or $U_{e3} = 0$ – probabilities take an effective 2f form.

(N.B.: $P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha)$)

Backup slides

Neutrino mixing schemes

I. Dirac case

$$-\mathcal{L}_{w+m} \;=\; \frac{g}{\sqrt{2}}(\bar{e}_L \gamma^\mu \, V_L^\dagger U_L \, \nu_L) \, W_\mu^- \;+\; \sum_{\alpha=1}^{n} m_{l\alpha} \bar{e}_\alpha e_\alpha \;+\; \sum_{i=1}^{n} m_i \bar{\nu}_i \nu_i \;+\; h.c.$$

Neutrino mixing schemes

I. Dirac case

$$-\mathcal{L}_{w+m} \;=\; \frac{g}{\sqrt{2}}(\bar{e}_L\gamma^\mu\, V_L^\dagger U_L\, \nu_L)\, W_\mu^- \;+\; \sum_{\alpha=1}^{n} m_{l\alpha}\bar{e}_\alpha e_\alpha \;+\; \sum_{i=1}^{n} m_i\bar{\nu}_i\nu_i \;+\; h.c.$$

$\diamondsuit$ $$V_L^\dagger U_L \;\equiv\; U\,; \qquad \nu_{\alpha L} \;=\sum_{i=1}^{n} U_{\alpha i}\,\nu_{iL} \qquad \Rightarrow \qquad |\nu_{\alpha L}\rangle \;=\sum_{i=1}^{n} U^*_{\alpha i}\,|\nu_{iL}\rangle$$

$$(\alpha \;=\; e\,,\;\mu\,,\;\tau, \qquad i \;=\; 1\,,\;2\,,\;3$$

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◇ $$P(\nu_\alpha \to \nu_\beta; L) = \left| \sum_{i=1}^{n} U_{\beta i}\, e^{-i \frac{\Delta m_{ij}^2}{2p} L}\, U_{\alpha i}^* \right|^2$$

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Osc. probability: the same expression

Neutrino mixing schemes

III. Dirac + Majorana mass term (n LH and k RH neutrinos)

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Index a can take $n+k$ values; denote collectively the first n of them with α and the last k with σ $\Rightarrow$

D + M mass term – contd.

Active and sterile LH neutrino fields in terms of LH components of mass eigenstates:

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Sterile - sterile neutrino oscillations:

$$P(\nu^c_{\sigma L} \to \nu^c_{\rho L}; L) = \left| \sum_{i=1}^{n+k} \mathcal{U}_{\rho i} \, e^{-i \frac{\Delta m^2_{ij}}{2p} L} \, \mathcal{U}^*_{\sigma i} \right|^2 .$$

2f oscillations: physical ranges of parameters

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$
$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

In general, $\theta \in [0, 2\pi]$. But: there are transformations that leave ν mixing formulas unchanged:

- $\theta \to \theta + \pi, \quad |\nu_1\rangle \to -|\nu_1\rangle, \quad |\nu_2\rangle \to -|\nu_2\rangle \quad \Rightarrow \quad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
- $\theta \to -\theta, \quad |\nu_2\rangle \to -|\nu_2\rangle, \quad |\nu_\mu\rangle \to -|\nu_\mu\rangle \quad \Rightarrow \quad \theta \in [0, \frac{\pi}{2}]$
- $\theta \to \frac{\pi}{2} - \theta, \quad |\nu_1\rangle \leftrightarrow |\nu_2\rangle, \quad |\nu_\mu\rangle \to -|\nu_\mu\rangle \quad \Rightarrow \quad \Delta m^2 \to -\Delta m^2$

One can always choose $\Delta m^2 > 0$ by choosing appropriately θ within $[0, \frac{\pi}{2}]$.

For vacuum oscillations: $P_{\rm tr}$, $P_{\rm surv}$ depend only on $\sin^2 2\theta$ $\Rightarrow$ one can choose θ to be in $[0, \frac{\pi}{4}]$. Not true for oscillations in matter!

Similar considerations in the 3f case: all $\theta_{ij} \in [0, \frac{\pi}{2}]$; $\delta_{\rm CP} \in [0, 2\pi]$.

The paradox of σ_E and σ_p

QM uncertainty relations: σ_p is related to the spatial localization of the production (detection) process, while σ_E to its time scale $\Rightarrow$ independent quantities.

On the other hand: Neutrinos propagating macroscopic distances are on the mass shell. For on-shell mass eigenstates $E^2 = p^2 + m_i^2$ means

$$E\sigma_E \;=\; p\sigma_p$$

How can this be understood?

The solution: At production, neutrinos are *not* on the mass shell. They go on shell only after they propagate $x \sim$ (a few)$\times$ De Broglie wavelengths. After that their energy and momentum get related by $E^2 = p^2 + m_i^2 \quad \Rightarrow \quad$ the larger uncertainty shrinks towards the smaller one to satisfy $E\sigma_E \;=\; p\sigma_p$.

On-shell relation between E and p allows to determine the less certain of the two through the more certain one, reducing the error of the latter.

What determines the length of ν w. packets?

The length of ν w. packets: $\sigma_x \sim 1/\sigma_p$. For propagating on-shell neutrinos:

$$\sigma_p \simeq \min\{\sigma_p^{\rm prod},\ (E/p)\sigma_E^{\rm prod}\} = \min\{\sigma_p^{\rm prod},\ (1/v_g)\sigma_E^{\rm prod}\}$$

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- If $T_S < \tau$ (τ – lifetime of the parent unstable particle) $\Rightarrow$ $\sigma_E \simeq T_S^{-1}$ (collisional broadening). Mom. uncertainty: $\sigma_p \simeq L_S^{-1}$.

  But: $L_S = v_S T_S \quad \Rightarrow \quad \sigma_E < \sigma_p$ (a consequence of $v_S < 1$)

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- If $T_S > \tau$ (quasi-free parent particle) $\Rightarrow$ $\sigma_E \simeq \tau^{-1} = \Gamma$.

  $\sigma_p \simeq [(p/E)\tau]^{-1} \simeq [(p/E)\sigma_E]^{-1}$, i.e. $\sigma_E \simeq (p/E)\sigma_p < \sigma_p$.

The length of ν w. packets – contd.

In both cases $\boxed{\sigma_E^{\rm prod} < \sigma_p^{\rm prod}}$ $\Leftarrow$ also when $\nu's$ are produced in collisions.

$$\Longrightarrow \qquad \sigma_{p\ \rm eff} \simeq \frac{\sigma_E}{v_g}\,, \qquad \boxed{\sigma_x \simeq \frac{v_g}{\sigma_E}}$$

In the stationary limit $(\sigma_E \to 0)$ one has $\sigma_{p\ \rm eff} \to 0$ even though σ_p is finite!
Therefore $\sigma_x \to \infty$ and so the coherence length $l_{\rm coh} \to \infty$
– a well known result.

Longitudinal *vs.* transversal w.p. dispersion

Spreading of the wave packets: consequence of the fact that the there is a spread of momenta inside of the wave packets and of the p-dependence of the group velocity.

$$v^i_{spr} \simeq \frac{\partial v_i}{\partial p^j}\sigma^j_p = \frac{1}{E}(\delta_{ij} - v_i v_j)\sigma^j_p = \frac{1}{E}[\sigma^i_p - v_i(\vec{v}\vec{\sigma_p})]$$

This gives

$$v^{\perp}_{spr.} = \frac{\sigma_p}{E}\,, \qquad v^{||}_{spr.} = \frac{\sigma_p}{E}(1-v^2) = \frac{\sigma_p}{E}\frac{m^2}{E^2}$$

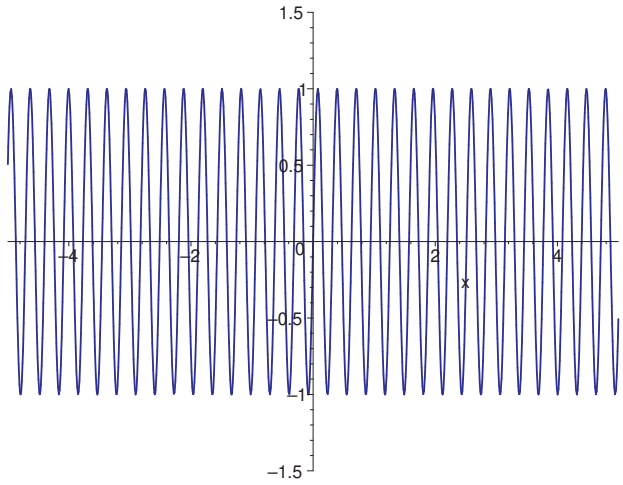
$t_{transv} \sim E/\sigma_p^2,\ t_{long.} \sim E^3/\sigma_p^2 m^2.$

The wave packet approach

In quantum theory propagating particles are described by wave packets!

– Finite extensions in space and time.

Plane waves: the wave function at time $t = 0$ $\quad \Psi_{\vec{p}_0}(\vec{x}) \;=\; e^{i\vec{p}_0\vec{x}}$



Wave packets: superpositions of plane waves with momenta in an interval of width σ_p around mom. p_0 $\Rightarrow$ constructive interference in a spatial interval of width σ_x around some point x_0 and destructive interference outside it.

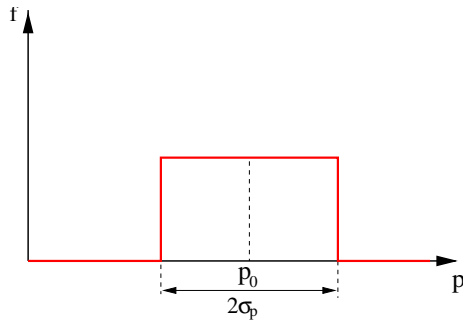
$$\sigma_x\,\sigma_p \geq 1/2 \quad - \quad \text{QM uncertainty relation}$$

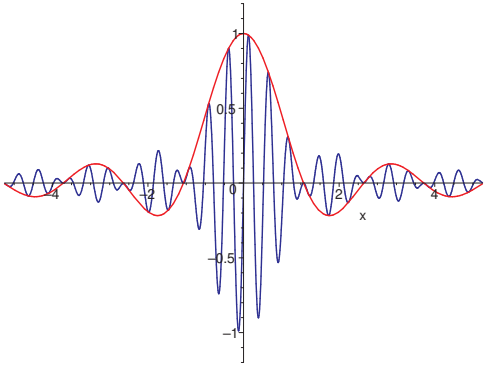
Wave packets

W. packet centered at $\vec{x}_0 = 0$ at time $t = 0$:

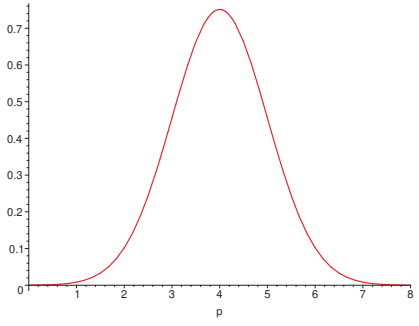
$$\Psi(\vec{x};\, \vec{p}_0, \sigma_{\vec{p}}) \;=\; \int \frac{d^3p}{(2\pi)^{3/2}}\, f(\vec{p} - \vec{p}_0)\, e^{i\vec{p}\,\vec{x}}$$

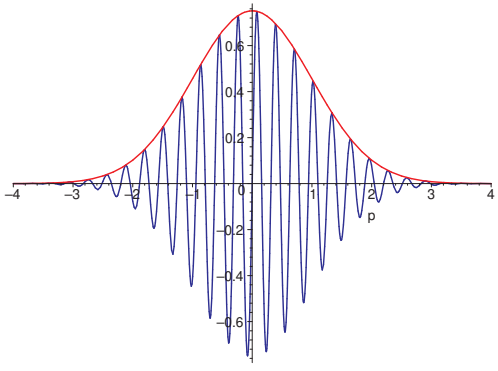
Rectangular mom. space w. packet:





Gaussian mom. space w. packet:





$\sigma_x \sigma_p \;=\; 1/2$ – minimum uncertainty packet

Propagating wave packets

Include time dependence:

$$\Psi(\vec{x},\, t) \;=\; \int \frac{d^3p}{(2\pi)^{3/2}}\, f(\vec{p}-\vec{p}_0)\, e^{i\vec{p}\vec{x}-iE(p)t}$$

Expand $E(p) \;=\; \sqrt{p^2+m^2}$ near $p \;=\; p_0$:

$$E(p) \;=\; E(p_0) \;+\; \left.\frac{\partial E(p)}{\partial \vec{p}}\right|_{\vec{p}_0} (\vec{p}-\vec{p}_0) \;+\; \frac{1}{2}\left.\frac{\partial^2 E(p)}{\partial \vec{p}^2}\right|_{\vec{p}_0} (\vec{p}-\vec{p}_0)^2 \;+\; \ldots$$

$$\vec{v}_g \;=\; \frac{\partial E(p)}{\partial \vec{p}} \;=\; \frac{\vec{p}}{E}\,, \qquad\qquad \alpha \;=\; \frac{\partial^2 E(p)}{\partial \vec{p}^2} \;=\; \frac{m^2}{E^2}$$

$$\boxed{\Psi(\vec{x},\, t) \;\simeq\; e^{i\vec{p}_0\vec{x}-iE(p_0)t} \int \frac{d^3p_1}{(2\pi)^{3/2}}\, f(\vec{p}_1)\, e^{i\vec{p}_1(\vec{x}-\vec{v}_g t)}} \qquad (\alpha \;\to\; 0)$$

Center of the wave packet: $\vec{x} - \vec{v}_g t = 0$

QM wave packet approach

The evolved produced state:

$$|\nu_\alpha^{\rm fl}(\vec{x},t)\rangle \;=\; \sum_i U^*_{\alpha i}\,|\nu_i^{\rm mass}(\vec{x},t)\rangle \;=\; \sum_i U^*_{\alpha i}\,\Psi_i^S(\vec{x},t)|\nu_i^{\rm mass}\rangle$$

The coordinate-space wave function of the ith mass eigenstate (w. packet):

$$\boxed{\Psi_i^S(\vec{x},t) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f_i^S(\vec{p})\, e^{i\vec{p}\vec{x}-iE_i(p)t}}$$

Momentum distribution function $f_i^S(\vec{p})$: sharp maximum at $\vec{p}=\vec{P}$ (width of the peak $\sigma_{pP} \ll P$).

$$E_i(p) \;=\; E_i(P) \;+\; \left.\frac{\partial E_i(p)}{\partial \vec{p}}\right|_{\vec{P}} (\vec{p}-\vec{P}) \;+\; \frac{1}{2}\left.\frac{\partial^2 E_i(p)}{\partial \vec{p}^2}\right|_{\vec{p}_0} (\vec{p}-\vec{P})^2 \;+\; \ldots$$

$$\vec{v}_i \;=\; \frac{\partial E_i(p)}{\partial \vec{p}} \;=\; \frac{\vec{p}}{E_i}\,, \qquad\qquad \alpha \;\equiv\; \frac{\partial^2 E_i(p)}{\partial \vec{p}^2} \;=\; \frac{m_i^2}{E_i^2}$$

Evolved neutrino state

$$\boxed{\Psi_i^S(\vec{x},\, t) \;\simeq\; e^{-iE_i(P)t+i\vec{P}\vec{x}}\, g_i^S(\vec{x}-\vec{v}_i t)} \qquad (\alpha \;\to\; 0)$$

$$g_i^S(\vec{x}-\vec{v}_i t) \;\equiv\; \int \frac{d^3p_1}{(2\pi)^3}\, f_i^S(\vec{p}_1)\, e^{i\vec{p}_1(\vec{x}-\vec{v}_g t)}$$

Center of the wave packet: $\vec{x}-\vec{v}_i t = 0$. Spatial length: $\sigma_{xP} \;\sim\; 1/\sigma_{pP}$ (g_i^S decreases quickly for $|\vec{x}-\vec{v}_i t| \gtrsim \sigma_{xP}$).

Detected state (centered at $\vec{x} = \vec{L}$):

$$|\nu_\beta^{\rm fl}(\vec{x})\rangle \;=\; \sum_k U_{\beta k}^*\, \Psi_k^D(\vec{x})|\nu_i^{\rm mass}\rangle$$

The coordinate-space wave function of the ith mass eigenstate (w. packet):

$$\boxed{\Psi_k^D(\vec{x}) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f_k^D(\vec{p})\, e^{i\vec{p}(\vec{x}-\vec{L})}}$$

Oscillation probability

Transition amplitude:

$$\mathcal{A}_{\alpha\beta}(T,\vec{L}) \;=\; \langle \nu_\beta^{\rm fl} | \nu_\alpha^{\rm fl}(T,\vec{L})\rangle \;=\; \sum_i U^*_{\alpha i} U_{\beta i}\, \mathcal{A}_i(T,\vec{L})$$

$$\boxed{\mathcal{A}_i(T,\vec{L}) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f_i^S(\vec{p})\, f_i^{D*}(\vec{p})\, e^{-iE_i(p)T + i\vec{p}\vec{L}}}$$

Strongly suppressed unless $|\vec{L} - \vec{v}_i T| \lesssim \sigma_x$. E.g., for Gaussian wave packets:

$$\mathcal{A}_i(T,\vec{L}) \;\propto\; \exp\left[-\frac{(\vec{L}-\vec{v}_i T)^2}{4\sigma_x^2}\right], \qquad \sigma_x^2 \;\equiv\; \sigma_{xP}^2 + \sigma_{xD}^2$$

Oscillation probability:

$$\diamondsuit \quad P(\nu_\alpha \to \nu_\beta; T, \vec{L}) \;=\; |\mathcal{A}_{\alpha\beta}|^2 = \sum_{i,k} U^*_{\alpha i} U_{\beta i} U_{\alpha k} U^*_{\beta k}\, \mathcal{A}_i(T,\vec{L}) \mathcal{A}_k^*(T,\vec{L})$$

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Neutrino emission and detection times are not measured (or not accurately measured) in most experiments $\Rightarrow$ integration over T:

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Here: $v \equiv \frac{v_i + v_k}{2}$, $\Delta v \equiv v_k - v_i$, $r_{i,k} \equiv \frac{v_{i,k}}{v}$, $N \equiv 1/[2E_i(P) 2E_k(P) v]$

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- $\tilde{I}_{ik}$ is also strongly suppressed unless $\Delta E_{ik}/v \ll \sigma_p$, i.e. $\Delta E_{ik} \ll \sigma_E$ – coherent production/detection condition

Oscillations and QM uncertainty relations

Neutrino oscillations – a QM interference phenomenon, owe their existence to QM uncertainty relations

Neutrino energy and momentum are characterized by uncertainties σ_E and σ_p related to the spatial localization and time scale of the production and detection processes. These uncertainties

- allow the emitted/absorbed neutrino state to be a coherent superposition of different mass eigenstates
- determine the size of the neutrino wave packets $\Rightarrow$ govern decoherence due to wave packet separation

σ_E – the effective energy uncertainty, dominated by the smaller one between the energy uncertainties at production and detection. Similarly for σ_p.

The normalization prescription

Oscillation probability calculated in QM w. packet approach is not automatically normalized ! Can be normalized "by hand" by imposing the unitarity condition:

$$\sum_\beta P_{\alpha\beta}(L) \;=\; 1\,.$$

This gives

$$\int dT|\mathcal{A}_i(L,T)|^2 = 1 \quad \Rightarrow \quad F_{ii} = N\int \frac{dp}{2\pi v}|f_i^S(p)|^2\,|f_i^D(p)|^2 \;=\; 1$$

– important for proving Lorentz invariance of the oscillation probability.

Depends on the overlap of $f_i^S(p)$ and $f_i^S(p)$ $\Rightarrow$ no independent normalization of the produced and detected neutrino wave function would do!

In QFT approach the correctly normalized $P_{\alpha\beta}(L)$ is automatically obtained and the meaning of the normalization procedure adopted in the w. packet approach clarified

Shortcomings of the QM w. packet approach

- Neutrino wave packet postulated rather than derived, widths estimated
- Production and detection processes are not considered
- Inadequate normalization procedure. Normalization “by hand” is unavoidable.

Advantage: simplicity

Lorentz invariance of oscillation probability

1. “Paradox” of neutrino w. packet length

For neutrino production in decays of unstable particles at rest (e.g. $\pi \to \mu\nu_\mu$):

$$\sigma_E \simeq \tau^{-1} = \Gamma_\pi \,, \qquad \sigma_x \simeq \frac{v_g}{\sigma_E} \simeq \frac{v_g}{\Gamma_\pi} \; (= v_g \tau)$$

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<u>The solution:</u> pion decay takes finite time. During the decay time the pion moves over distance $l = u\tau'$ (“chases” the neutrino if $u > 0$).

$$\sigma'_x \simeq v'_g/\Gamma' - l = v'_g\tau' - u\tau' = (v'_g - u)\gamma_u\tau = \frac{v_g\tau}{\gamma_u(1 + v_g u)} \,,$$

[the relativ. law of addition of velocities: $v'_g = (v_g + u)/(1 + v_g u)$].

Lorentz invariance issues – contd.

That is

$$\sigma'_x = \frac{\sigma_x}{\gamma_u(1 + v_g u)}$$

For relativistic neutrinos $v_g \approx v'_g \approx 1 \quad \Rightarrow$

$$\sigma'_x = \sigma_x \sqrt{\frac{1-u}{1+u}}$$

$\Rightarrow$ when the pion is boosted in the direction of neutrino emission $(u > 0)$ the neutrino wave packet gets contracted; when it is boosted in the opposite direction $(u < 0)$ – the wave packet gets dilated.

Lorentz invariance issues – contd.

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How can we see Lorentz invariance of the standard formula for the oscillation probability ? P_{ab} depends on L/p (contains factors $\exp[-i\frac{\Delta m^2_{ik}}{2p}L]$). Is L/p Lorentz invariant?

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$$L' = \gamma_u(L + ut)\,, \qquad t' = \gamma_u(t + uL)\,,$$
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The stand. osc. formula results when (i) production and detection and (ii) propagation are coherent; for neutrinos from conventional sources (i) implies $\sigma_x \ll l_{\rm osc}$. $\Rightarrow$ one can consider neutrinos pointlike and set $L = v_g t$. $\Rightarrow$ $L' = \gamma_u L(1 + u/v_g)$.

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$\Rightarrow$ $$\boxed{L'/p' \;=\; L/p}$$

Lorentz invariance issues – contd.

A more general argument (applies also to Mössbauer neutrinos which are not pointlike): Consider the phase difference

$$\diamondsuit \qquad \Delta\phi \;=\; -\frac{1}{v_g}(L \;-\; v_g\, t)\Delta E \;+\; \frac{\Delta m^2}{2p}\, L$$

– a Lorentz invariant quantity, though the two terms are in not in general separately Lorentz invariant.

<u>But:</u> If the 1st term is negligible in all Lorentz frames, the second term is Lorentz invariant by itself $\Rightarrow$ L/p is Lorentz invariant.

The 1st term can be neglected when the production/detection coherence conditions are satisfied. In particular, it vanishes in the limit of pointlike neutrinos $L = v_g t$. N.B.:

$$L' - v_g' t' \;= \gamma_u \left[(L + ut) - \frac{v_g + u}{1 + v_g u}(t + uL) \right] \;=\; \frac{L - v_g t}{\gamma_u (1 + v_g u)}\,,$$

i.e. the condition $L = v_g t$ is Lorentz invariant. MB neutrinos: $\Delta E \simeq 0$.

Lorentz invariance issues – contd.

The oscillation probability must be Lorentz invariant even when the coherence conditions are not satisfied !
Lorentz invariance is enforced by the normalization condition.

$$P_{ab}(L) \;=\; \sum_{i,k} U_{ai} U^*_{bi} U^*_{ak} U_{bk}\, I_{ik}(L)\,, \qquad \text{where}$$

$$I_{ik}(L) \;\equiv\; \int dT\, \mathcal{A}_i(L,T) \mathcal{A}^*_k(L,T) e^{-i\Delta\phi_{ik}}$$

From the norm. cond. $\int dT\, |\mathcal{A}_i(L,T)|^2 \;=\; 1 \quad \Rightarrow$

$$|\mathcal{A}_i|^2 dt \;=\; inv. \quad \Rightarrow \quad |\mathcal{A}_i||\mathcal{A}_k| dt \;=\; inv. \quad \Rightarrow \quad \mathcal{A}_i \mathcal{A}^*_k dT \;=\; inv.$$

The phase difference $\Delta\phi_{ik} = \Delta E_{ik} T - \Delta p_{ik} L$ is also Lorentz invariant $\Rightarrow$ so is $I_{ik}(L)$, and consequently $P_{ab}(L)$.

Coherence and Lorentz invariance

Consider coherent production conditions $\Delta E \ll \sigma_E$, $\Delta p \ll \sigma_p$ for ν born in $\pi^\pm$ decays in the rest frame of ν_2 (EA, 1703.08169):

$$\frac{|\Delta E'|}{\sigma'_E} \simeq \frac{\Delta m^2}{2m_2}\frac{\gamma_u}{\Gamma_\pi} \simeq \frac{\Delta m^2}{2E\Gamma_\pi}\gamma_u^2\,, \qquad \frac{|\Delta p'|}{\sigma'_{p\mathrm{min}}} \simeq \frac{\Delta m^2}{2m_2}v_{g2}\frac{\gamma_u}{\Gamma_\pi} \simeq \frac{\Delta m^2}{2E\Gamma_\pi}\gamma_u^2$$

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Lorentz factor $\gamma_u = E/m_2 \gg 1 \;\Rightarrow\;$ the conditions $\Delta E' \ll \sigma'_E$, $\Delta p' \ll \sigma'_p$ can be violated for small enough m_2. Moreover, for non-rel. neutrinos quite generally $\Delta E \sim \bar{E} \gtrsim \sigma_E$!

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Resolution: the conditions $\Delta E \ll \sigma_E$, $\Delta p \ll \sigma_p$ are <u>not</u> Lorentz invarint. They follow form the Lorentz-inv. coherent production condition

$$|\Delta E \cdot \delta t - \Delta p \cdot \delta x| \ll 1$$

only assuming that the two terms on the LHS do not (approximately) cancel each other and are separately small.

In reality: Lorentz transformations with $u = -v_{g2} \simeq -1$ give

$$\delta t' = \gamma_u(\delta t + u\delta x) \simeq \gamma_u(\delta t - \delta x)\,, \qquad \delta x' = \gamma_u(\delta x + u\delta t) \simeq \gamma_u(\delta x - \delta t)\,,$$

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In the rest frame of ν_2 the two terms in $\delta\phi'_{osc}$ approximately cancel each other:

$$\delta\phi'_{osc} = \Delta E' \cdot \delta t' - \Delta p' \cdot \delta x' \simeq \Delta E' \cdot (\delta t' + \delta x') \simeq 0\,.$$

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– no enhancement of $\delta\phi'_{osc}$ actually occurs!
More accurate calculation (taking into account the small deviation of $u = -v_{g2}$ from -1):

$$\delta\phi'_{osc} = \delta\phi_{osc} \ll 1.$$

Conditions $\Delta E \ll \sigma_E$, $\Delta p \ll \sigma_p$ are valid only in the frames where the neutrino source is at rest or is slowly moving. Should be used with caution! Cannot be automatically extrapolated from one Lorentz frame to another.